Exponential martingales and changes of measure for counting processes

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Agenda

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2. Exponential martingales and changes of measure
3. Examples
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Modeling counting processes

A classical method for modeling discrete events in continuous time is through counting processes.

A statistical model for a counting process with intensity consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
- A nonexplosive point process N on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
- A parametrized family $(\mu_\theta)_{\theta\in\Theta}$ of intensities.
- A corresponding family of probability measures P_θ such that under P_θ, N is a nonexplosive counting process with intensity μ_θ.

Problem. Given a family $(\mu_\theta)_{\theta\in\Theta}$, does there exist a statistical model corresponding to this family of candidate intensities? This is not a vacuous question, as many candidate intensities yield explosion.

Modeling counting processes

Solution approach on canonical spaces. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be the space of nonexplosive point process trajectories endowed with the canonical σ-algebra and filtration, let $N : \Omega \to \Omega$ be the identity and let P be such that N is a homogeneous Poisson process.

Jacobsen (2005) gives sufficient criteria on μ_θ to ensure that there exists a probability measure P_θ equivalent to P such that under P_θ, N has intensity μ_θ.

This yields the existence of nonexplosive point processes with intensity μ_θ and yields the existence of the statistical model.

Modeling counting processes

Benefits of the canonical setting:

- Precise expressions for the likelihood in terms of the waiting time distributions of the point process with intensity μ_θ.
- Coupling arguments may be used to analyze non-explosion.

Drawbacks of the canonical setting:

- Only intensities depending on N are covered.
- Arguments are often based on very technical manipulations of the canonical space and various conditional distributions, instead of for example modern martingale theory.

Alternative approach. Consider a general filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and formulate all issues in terms of martingales.

Modeling counting processes

A general problem statement. Assume given:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.
- A positive, predictable and locally bounded intensity process λ.
- A point process N with intensity λ.
- A parametrized family $(\mu_\theta)_{\theta \in \Theta}$ of intensities.

We seek: Sufficient criteria on μ_θ to ensure the existence of a probability measure P_θ equivalent to P such that under P_θ, N has intensity μ_θ.

As corollaries, we obtain: Explicit expressions for the likelihood, criteria for existence of point processes with various intensities (corresponding to criteria for nonexplosion).

Exponential martingales and changes of measure

Definition. We say that a d-dimensional nonexplosive point process N has intensity λ if $N^i_t - \int_0^t \lambda^i_s \, \mathrm{d}s$ is a local martingale, $i \leq d$.

From now on, assume given:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions.
- Positive, predictable and locally bounded d-dimensional λ, μ.
- A d-dimensional point process N with intensity λ.

For any semimartingale X with $\Delta X > -1$, we define

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X^c]_t + \sum_{0<s\leq t} \log(1 + \Delta X_s) - \Delta X_s \right).$$

$\mathcal{E}(X)$ is the Doléans-Dade exponential of X. If X is a local martingale, so is $\mathcal{E}(X)$.

Exponential martingales and changes of measure

We define:

- $M_t^i = N_t^i - \int_0^t \lambda_s^i \,\mathrm{d}s$.
- $\gamma_t^i = \mu_t^i (\lambda^i)_t^{-1}$.
- $H_t^i = \gamma_t^i - 1$.
- $H \cdot M = \sum_{i=1}^d \int_0^t H_s^i \,\mathrm{d}M_s^i$.

Lemma. Assume that $\mathcal{E}(H \cdot M)$ is a martingale. Let $t \geq 0$. With Q_t being the measure with Radon-Nikodym derivative $\mathcal{E}(H \cdot M)_t$ with respect to P, N is a counting process under Q_t with intensity $1_{[0,t]}\mu + 1_{(t,\infty)}\lambda$.

Conclusion. In order to obtain the existence of the desired equivalent probability measures, we need criteria for the martingale property of $\mathcal{E}(H \cdot M)$.

Exponential martingales and changes of measure

Theorem. Assume that there is $\varepsilon > 0$ such that whenever $0 \leq u \leq t$ with $|t-u| \leq \varepsilon$, one of the following two conditions are satisfied:

$$E \exp\left(\sum_{i=1}^{d} \int_u^t (\gamma_s^i \log \gamma_s^i - (\gamma_s^i - 1))\lambda_s^i \,\mathrm{d}s\right) < \infty \quad \text{or}$$

$$E \exp\left(\sum_{i=1}^{d} \int_u^t \lambda_s^i \,\mathrm{d}s\right) < \infty \quad \text{and} \quad E \exp\left(\sum_{i=1}^{d} \int_u^t \log_+ \gamma_s^i \,\mathrm{d}N_s^i\right) < \infty,$$

where $\log_+ x = \max\{\log x, 0\}$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Exponential martingales and changes of measure

Corollary. Let $\lambda = 1$. Assume that there is $\varepsilon > 0$ such that whenever $0 \leq u \leq t$ with $|t - u| \leq \varepsilon$, one of the following two conditions are satisfied:

$$E \exp\left(\sum_{i=1}^{d} \int_u^t \mu_s^i \log_+ \mu_s^i \, \mathrm{d}s\right) < \infty \quad \text{or}$$
$$E \exp\left(\sum_{i=1}^{d} \int_u^t \log_+ \mu_s^i \, \mathrm{d}N_s^i\right) < \infty,$$

where $\log_+ x = \max\{\log x, 0\}$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Exponential martingales and changes of measure

Outline of proof:

1. Argue that it suffices to show that $\mathcal{E}((H \cdot M)^t - (H \cdot M)^u)$ is a martingale for $|t - u| \leq \varepsilon$.
2. Decompose μ into large and small parts and show a related decomposition for exponential martingales.
3. Apply two theorems of Lépingle & Mémin (1978) to the obtain the result.

Examples

Example 1. Let $\mu_t^i \leq \alpha + \beta \sum_{j=1}^d N_{t-}^j$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Example 1 shows that we may recover the classical affine criteria for non-explosion from the canonical case in the case of a general filtered space. This also extends the criterion from Gjessing et al. (2010) from an "$\mathcal{L}^p$"-criterion, $p > 1$, to an "$\mathcal{L}^p$"-criterion, $p \geq 1$.

Outline of proof: To use the first moment condition, use that $E \exp(\varepsilon X \log X)$ is finite when X is Poisson distributed and $0 < \varepsilon < 1$, choose $\varepsilon > 0$ such that $4\beta\varepsilon d < 1$. To use the second moment condition, use a Markov argument and that Poisson distributions have moments of all orders, choose $\varepsilon > 0$ such that $\beta\varepsilon d < 1$.

Examples

Example 2. Consider $A : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{R}^d$, $B : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{M}(d,d)$ and $\sigma : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{M}(d,d)$. Assume that $A(\eta,\cdot)$, $B(\eta,\cdot)$ and $\sigma(\eta,\cdot)$ are continuous and bounded for $\eta \in \mathbb{N}_0^d$. Assume that σ is positive definite. Assume that for $\eta \in \mathbb{N}_0^d$, there is $\delta, c > 0$ such that

$$\sup_{t\geq 0} \|A(\eta,t)\|_2 \leq c\|\eta\|_1^{1-\delta}$$

$$\sup_{t\geq 0} \|\sigma(\eta,t)\|_2 \leq c\|\eta\|_1^{(1-\delta)/2}$$

$$\sup_{t\geq 0} \|B(\eta,t)\|_2 \leq c.$$

Examples

Example 2, contined. Let X be a solution to

$$\mathrm{d}X_t = (A(N_t, Z_t) + B(N_t, Z_t)X_t)\,\mathrm{d}t + \sigma(N_t, Z_t)\,\mathrm{d}W_t,$$

where W is a d-dimensional $(\mathcal{F}_t)$ Brownian motion and $Z_t^i = t - T^i_{N_t^i}$, where T_n^i is the n'th event time of N^i. Let $\phi : \mathbb{R}^d \to \mathbb{R}^d_+$ be Lipschitz. Assume that $\phi(x)^i \neq 0$ for $x_i \neq 0$. Put $\mu_t = \phi(X_t)$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Example 2 shows that we can use our results to construct counting processes where the intensity is driven by a SDE whose coefficients vary according to the history of the counting process.

Examples

Outline of proof: Note that conditionally on N, the intensity has the distribution of a Gaussian process. Apply bounds for $E\exp(c\|Z\|_2^{1+\varepsilon})$, with Z d-dimensionally Gaussian and $0 < \varepsilon < 1$, to obtain a bound for the conditional expectation

$$E\left(\exp\left(t\sum_{i=1}^{d}\mu_s^i\log_+\mu_s^i\right)\middle| N\right).$$

Use this to obtain a bound of the unconditional expectation varying continuously in s, $0 \le s \le t$. Apply Jensen's inequality and further estimates to obtain the result.

Examples

Example 3. Let $\phi_i : \mathbb{R} \to (0, \infty)$ for $i \leq d$ and $h_{ij} : \mathbb{R}_+ \to \mathbb{R}$ for $i, j \leq d$. Define

$$\mu_t^i = \phi_i \left(\sum_{j=1}^{d} \int_0^{t-} h_{ij}(t-s) \, \mathrm{d}N_s^j \right).$$

Assume that ϕ^i is Borel measurable, that $\phi_i(x) \leq |x|$ and that h_{ij} is bounded. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Example 3 is an example of a sufficient criterion for non-explosion for multidimensional Hawkes processes.

Open problems

Questions not yet answered:

- Can the requirement that μ be positive be dropped if we require only absolute continuity instead of equivalence between measures?
- Can Example 2 be extended to the case where $\delta = 0$?

References

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