An introduction to stochastic integration with respect to general semimartingales

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Preface

This monograph concerns itself with the theory of continuous-time martingales and the theory of stochastic integration with respect to general semimartingales. One primary question in the theory of stochastic integration is the following: For what integrators X and integrands H can the integral

$$\int_0^t H_s \,\mathrm{d}X_s$$

be defined in a sensible manner? As many stochastic processes of interest as integrators, for example the Brownian motion, have paths which almost surely are of infinite variation, this integral cannot immediately be defined by reference to ordinary Lebesgue integration theory. Therefore, another approach is necessary. It turns out that the proper notion of integrator X is that of a semimartingale, which is the sum of a local martingale and a process with paths of finite variation, while the natural measurability requirement on the integrand H is that it be measurable with respect to the predictable σ-algebra, which is the σ-algebra on $\mathbb{R}_+ \times \Omega$ generated by the left-continuous and adapted processes. That these are the correct answers is by no means self-evident, and builds on the deep insights of many people.

Once the stochastic integral has been constructed, its properties may be investigated, leading for example to Itô's formula, the change-of-variables theorem for stochastic calculus, the entry point for making stochastic calculus an operational theory applicable to both other fields of probability theory and to practical statistical modeling.

Several introductory accounts of the theory of stochastic integration exist. One of the first complete accounts is given in Dellacherie & Meyer (1978). Good alternative books are He et al. (1992), Rogers & Williams (2000b), Kallenberg (2002), Jacod & Shiryaev (2003) and Protter (2005), for example. The purpose of this monograph is to apply certain techniques to simplify the theory so as to present a very direct path to the fundamentals of martingale theory, the general theory of processes and the stochastic integral.

As prerequisites, the reader is assumed to have a reasonable grasp of basic analysis, measure theory and discrete-time martingale theory, as can be obtained through the books Carothers (2000), Ash (2000) and Rogers & Williams (2000a). Familiarity with continuous-time stochastic processes and the theory of stochastic integration with respect to continuous semi-martingales as in Karatzas & Shreve (1991) is also beneficial, though not strictly required.

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Chapter 1

Continuous-time stochastic processes

In this chapter, we develop the basic results of stochastic processes in continuous time, covering mostly some basic measurability results as well as the theory of continuous-time martingales. The results of this chapter form an essential part of the fundament for the theory to be developed in the following chapters.

In Section 1.1, we concern ourselves with the measurability properties of stochastic processes in continuous time, introducing the most frequently occurring path properties, as well as regularity properties such as being measurable, adapted and progressive. We introduce stopping times and prove that several classes of frequently occurring random variables are stopping times.

Section 1.2 concerns itself with continuous-time martingales. Applying the results from discrete-time martingale theory, we show the supermartingale convergence theorem, the optional sampling theorem and related results.

Section 1.3 introduces square-integrable martingales. Using the results developed in this section, we prove the existence of the quadratic variation process for bounded martingales, a particular case of the much more general construction to be carried out in Chapter 3. The construction made in this section, however, is remarkable for requiring almost no advanced theory.

In Section 1.4, we introduce the Lebesgue integral process with respect to a process with paths of finite variation, and ensure that we may always obtain a version of such an integral process satisfying certain measurability properties. The results of this section are elementary, yet will be important for the development of the theory of local martingales in Chapter 3, as well as for the development of the stochastic integral in Chapter 4.

1.1 Measurability and stopping times

We begin by reviewing basic results on continuous-time stochastic processes. We will work in the context of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Here, Ω denotes some set, $\mathcal{F}$ is a σ-algebra on Ω, P is a probability measure on $(\Omega, \mathcal{F})$ and $(\mathcal{F}_t)_{t\geq 0}$ is a family of σ-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $0 \leq s \leq t$ and such that $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \geq 0$. We refer to $(\mathcal{F}_t)_{t\geq 0}$ as the filtration of the probability space. We will require that the filtered probability space satisfies certain regularity properties given in the following definition. Recall that a P null set of $\mathcal{F}$ is a set $F \subseteq \Omega$ with the property that there exists $G \in \mathcal{F}$ with $P(G) = 0$ such that $F \subseteq G$.

Definition 1.1.1. *A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is said to satisfy the usual conditions if it holds that the filtration is right-continuous in the sense that $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \geq 0$, and for all $t \geq 0$, $\mathcal{F}_t$ contains all P null sets of $\mathcal{F}$. In particular, all P null sets of $\mathcal{F}$ are $\mathcal{F}$ measurable.*

We will always assume that the usual conditions hold.

We define $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$. A stochastic process is a family $(X_t)_{t\geq 0}$ of random variables taking values in $\mathbb{R}$. The sample paths of the stochastic process X are the functions $X(\omega)$ for $\omega \in \Omega$.

In the following, $\mathcal{B}$ denotes the Borel-σ-algebra on $\mathbb{R}$. We put $\mathbb{R}_+ = [0, \infty)$ and let $\mathcal{B}_+$ denote the Borel-σ-algebra on $\mathbb{R}_+$, and we let $\mathcal{B}_t$ denote the Borel-σ-algebra on $[0, t]$. We say that two processes X and Y are versions if $P(X_t = Y_t) = 1$ for all $t \geq 0$. In this case, we say that Y is a version of X and vice versa. We say that two processes X and Y are indistinguishable if their sample paths are almost surely equal, in the sense that the set where X and Y are not equal is a null set, meaning that the set $\{\omega \in \Omega \mid \exists\, t \geq 0 : X_t(\omega) \neq Y_t(\omega)\}$ is a null set.

We then say that X is a modification of Y and vice versa. We call a process evanescent if it is indistinguishable from the zero process, and we call a set $A \in \mathcal{B}_+ \otimes \mathcal{F}$ evanescent if the process 1_A is evanescent. We say that a result holds up to evanescence, or up to indistinguishability, if it holds except perhaps on an evanescent set. We have the following three measurability concepts for stochastic processes.

Definition 1.1.2. *Let X be a stochastic process. We say that X is adapted if X_t is $\mathcal{F}_t$ measurable for all $t \geq 0$. We say that X is measurable if $(t, \omega) \mapsto X_t(\omega)$ is $\mathcal{B}_+ \otimes \mathcal{F}$ measurable. We say that X is progressive if, for $t \geq 0$, the restriction of $(t, \omega) \mapsto X_t(\omega)$ to $[0, t] \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable.*

If a process X has sample paths which are all continuous, we say that X is continuous, and likewise for right-continuity, left-continuity, having right-continuous paths with left limits and having left-continuous paths with right limits. We refer to a mapping with right-continuous paths and left limits as a càdlàg mapping, and we refer to a mapping with left-continuous paths and right limits as a càglàd mapping. We refer to a process with càdlàg paths as a càdlàg process and we refer to a process with càglàd paths as a càglàd process. Note that we always require that path properties hold for all paths and not only almost surely.

For a càdlàg process X, the left limit $\lim_{s \to t-} X_s$ is well-defined for all $t > 0$ and is denoted by X_{t-}. We use the convention that $X_{0-} = X_0$. Writing $(X_-)_t = X_{t-}$, the process X_- is then well-defined on all of $\mathbb{R}_+$. We also introduce $\Delta X = X - X_-$ and refer to ΔX as the jump process of X. Note that by our conventions, ΔX always has initial value zero, so that there is no jump at the timepoint zero. Also, we define $\Delta X_\infty = 0$. For any càdlàg process, ΔX_t is then defined for all $t \in [0, \infty]$. Next, we introduce the progressive σ-algebra Σ^π and consider its basic properties.

Lemma 1.1.3. *Let Σ^π be the family of sets $A \in \mathcal{B}_+ \otimes \mathcal{F}$ such that $A \cap [0, t] \times \Omega \in \mathcal{B}_t \otimes \mathcal{F}_t$ for all $t \in \mathbb{R}_+$. Then Σ^π is a σ-algebra, and a process X is progressively measurable if and only if it is Σ^π-measurable.*

Proof. We first show that Σ^π is a σ-algebra. It holds that Σ^π contains $\mathbb{R}_+ \times \Omega$. If $A \in \Sigma^\pi$, we have $A \cap [0, t] \times \Omega \in \mathcal{B}_t \otimes \mathcal{F}_t$ for all $t \geq 0$. As $A^c \cap [0, t] \times \Omega = ([0, t] \times \Omega) \setminus (A \cap [0, t] \times \Omega)$, $A^c \cap [0, t] \times \Omega$ is the complement of $A \cap [0, t] \times \Omega$ relative to $[0, t] \times \Omega$. Therefore, as $\mathcal{B}_t \otimes \mathcal{F}_t$ is stable under complements, we find that $A^c \cap [0, t] \times \Omega$ is in $\mathcal{B}_t \otimes \mathcal{F}_t$ as well for all $t \geq 0$. Thus, Σ^π is stable under taking complements. Analogously, we find that Σ^π is stable under taking countable unions, and so Σ^π is a σ-algebra. As regards the statement on measurability, we first note for any $A \in \mathcal{B}$ the equality

$$\{(s,\omega) \in \mathbb{R}_+ \times \Omega \mid X(s,\omega) \in A\} \cap [0,t] \times \Omega = \{(s,\omega) \in [0,t] \times \Omega \mid X_{|[0,t]\times\Omega}(s,\omega) \in A\}.$$

Now assume that X is progressive. Fix a set $A \in \mathcal{B}$. From the above, we then obtain $\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid X(t,\omega) \in A\} \cap [0,t] \times \Omega \in \mathcal{B}_t \otimes \mathcal{F}_t$, so that X is Σ^π measurable. In order to obtain the converse implication, assume that X is Σ^π measurable. The above then shows $\{(s,\omega) \in [0,t] \times \Omega \mid X_{|[0,t]\times\Omega}(s,\omega) \in A\} \in \mathcal{B}_t \otimes \mathcal{F}_t$. Thus, being progressive is equivalent to being Σ^π measurable. □

Lemma 1.1.4. *Let X be left-continuous. Define $X^n_t = \sum_{k=0}^\infty X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(t)$. Then X^n converges pointwise to X.*

Proof. Fix $\omega \in \Omega$. It is immediate that $X_0(\omega) = \lim_n X^n_0(\omega)$, so it suffices to prove that $X_t(\omega) = \lim_n X^n_t(\omega)$ for $t > 0$. Fix $t > 0$. Let $\varepsilon > 0$, and take $\delta > 0$ with $\delta \le t$ such that $|X_t(\omega) - X_s(\omega)| \le \varepsilon$ whenever $s \in [t-\delta, t]$. Take n so large that $2^{-n} \le \delta$. We then obtain $X^n_t(\omega) = X_{k2^{-n}}(\omega)$ for some k with $k2^{-n} \le t < (k+1)2^{-n}$, so $k2^{-n} \in [t - 2^{-n}, t] \subseteq [t-\delta, t]$ and thus $|X^n_t(\omega) - X_t(\omega)| \le \varepsilon$. We conclude that $X_t(\omega) = \lim_n X^n_t(\omega)$, as desired. □

Lemma 1.1.5. *Let X be right-continuous. Fix $t \ge 0$. For $s \ge 0$, define a process X^n by putting $X^n_s = X_0 1_{\{0\}}(s) + \sum_{k=0}^{2^n-1} X_{t(k+1)2^{-n}} 1_{(tk2^{-n},t(k+1)2^{-n}]}(s)$. Then $X_s = \lim_n X^n_s$ pointwise for $0 \le s \le t$.*

Proof. Fix $\omega \in \Omega$ and $t \ge 0$. It is immediate that $X_0(\omega) = \lim_n X^n_0(\omega)$, so it suffices to consider the case $0 < s \le t$ and prove $X_s(\omega) = \lim_n X^n_s(\omega)$. To this end, let $\varepsilon > 0$, and take $\delta > 0$ such that $|X_s(\omega) - X_u(\omega)| \le \varepsilon$ whenever $u \in [s, s+\delta]$. Pick n so large that $t2^{-n} \le \delta$. Then $X^n_s(\omega) = X_{t(k+1)2^{-n}}(\omega)$ for some $k \le 2^n - 1$ with $tk2^{-n} < s \le t(k+1)2^{-n}$. This yields $t(k+1)2^{-n} \in [s, s+t2^{-n}] \subseteq [s, s+\delta]$ and thus $|X^n_s(\omega) - X_s(\omega)| \le \varepsilon$. From this, we obtain $X_s(\omega) = \lim_n X^n_s(\omega)$, as desired. □

Lemma 1.1.6. *Let X be an adapted process. Assume that X either is left-continuous or right-continuous. Then X is progressive.*

Proof. First consider the case where X is adapted and has left-continuous paths. By Lemma 1.1.4, $X_t = \lim_n X^n_t$ pointwise, where $X^n_t = \sum_{k=0}^\infty X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(t)$. Therefore, using the result from Lemma 1.1.3 that being progressive means measurability with respect to the σ-algebra Σ^π, we find that in order to show the result, it suffices to show that the process $t \mapsto X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(t)$ is progressive for any $n \ge 1$ and $k \ge 0$, since in this case, X inherits measurability with respect to Σ^π as a limit of Σ^π measurable maps. In order to show that $t \mapsto X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(t)$ is progressive, let $A \in \mathcal{B}$ with $0 \notin A$. For any $t \geq 0$, we then have

$$\begin{aligned} & \{(s,\omega) \in [0,t] \times \Omega \mid X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(s) \in A\} \\ = \quad & [k2^{-n},(k+1)2^{-n}) \cap [0,t] \times (X_{k2^{-n}} \in A). \end{aligned}$$

If $k2^{-n} > t$, this is empty and so in $\mathcal{B}_t \otimes \mathcal{F}_t$, and if $k2^{-n} \leq t$, this is in $\mathcal{B}_t \otimes \mathcal{F}_t$ as a product of a set in $\mathcal{B}_t$ and a set in $\mathcal{F}_t$. Thus, in both cases, we obtain an element of $\mathcal{B}_t \otimes \mathcal{F}_t$, and from this we conclude that the restriction of $t \mapsto X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(t)$ to $[0,t] \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable, demonstrating that the process is progressive. This shows that X is progressive.

Next, consider the case where X is adapted and has right-continuous paths. In this case, we fix $t \geq 0$ and define, for $0 \leq s \leq t$, $X^n_s = X_0 1_{\{0\}}(s) + \sum_{k=0}^{2^n-1} X_{t(k+1)2^{-n}} 1_{(tk2^{-n},t(k+1)2^{-n}]}(s)$. By Lemma 1.1.5, $X_s = \lim_n X^n_s$ pointwise for $0 \leq s \leq t$. Also, each term in the sum defining X^n is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable, and therefore, X^n is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable. As a consequence, the restriction of X to $[0,t] \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable, and so X is progressive. This concludes the proof. □

Lemma 1.1.7. *Let X be right-continuous or left-continuous. If X_t is almost surely zero for all $t \geq 0$, X is evanescent.*

Proof. We claim that $\{\omega \in \Omega \mid \ \forall\ t \geq 0 : X_t(\omega) = 0\} = \cap_{q \in \mathbb{Q}_+} \{\omega \in \Omega \mid X_q(\omega) = 0\}$ in both cases. The inclusion towards the right is immediate. In order to show the inclusion towards the left, assume that ω is such that $X_q(\omega)$ is zero for all $q \in \mathbb{Q}_+$. Let $t \in \mathbb{R}_+$. If X is left-continuous, we can use the density properties of $\mathbb{Q}_+$ in relation to $\mathbb{R}_+$ to obtain a sequence (q_n) in $\mathbb{Q}_+$ converging upwards t, yielding $X_t(\omega) = \lim_n X_{q_n}(\omega) = 0$. If X is right-continuous, we may instead pick a sequence converging downwards to t and again obtain that $X_t(\omega) = 0$. This proves the inclusion towards the left. Now, as a countable intersection of almost sure sets again is an almost sure set, we find that $\cap_{q \in \mathbb{Q}_+} \{\omega \in \Omega \mid X_q(\omega) = 0\}$ is an almost sure set. Therefore, $\{\omega \in \Omega \mid \ \forall\ t \geq 0 : X_t(\omega) = 0\}$ is an almost sure set, showing that X is evanescent. □

Lemma 1.1.8. *Let X be progressive. Then X is measurable and adapted.*

Proof. That X is measurable follows from Lemma 1.1.3. To show that X is adapted, recall that when X is progressive, the restriction of X to $[0,t] \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$-measurable, and therefore $\omega \mapsto X_t(\omega)$ is $\mathcal{F}_t$-measurable. □

Next, we define stopping times in continuous time and consider their interplay with measurability on $\mathbb{R}_+ \times \Omega$. A stopping time is a random variable $T : \Omega \to [0, \infty]$ such that $(T \leq t) \in \mathcal{F}_t$ for any $t \geq 0$. We say that T is finite if T maps into $\mathbb{R}_+$. We say that T is bounded if T maps into a bounded subset of $\mathbb{R}_+$. If X is a stochastic process and T is a stopping time, we denote by X^T the process $X^T_t = X_{T \wedge t}$ and call X^T the process stopped at T. Furthermore, we define the stopping-time σ-algebra $\mathcal{F}_T$ of events determined at T by putting $\mathcal{F}_T = \{A \in \mathcal{F} \mid A \cap (T \leq t) \in \mathcal{F}_t \text{ for all } t \geq 0\}$. It is immediate that $\mathcal{F}_T$ is a σ-algebra, and if T is constant, the stopping time σ-algebra is the same as the filtration σ-algebra, in the sense that $\{A \in \mathcal{F} \mid A \cap (s \leq t) \in \mathcal{F}_t \text{ for all } t \geq 0\} = \mathcal{F}_s$.

Lemma 1.1.9. *The following statements hold about stopping times:*

(1). Any constant in $[0, \infty]$ *is a stopping time.*

(2). A nonnegative variable T *is a stopping time if and only if* $(T < t) \in \mathcal{F}_t$ *for* $t \geq 0$.

(3). If S *and* T *are stopping times, so are* $S \wedge T$, $S \vee T$ *and* $S + T$.

(4). If T *is a stopping time and* $F \in \mathcal{F}_T$, *then* $T_F = T 1_F + \infty 1_{F^c}$ *also is a stopping time.*

(5). If $S \leq T$, *then* $\mathcal{F}_S \subseteq \mathcal{F}_T$.

Proof. **Proof of (1).** Let c be a constant in $\mathbb{R}_+$. Then $(c \leq t)$ is either $\emptyset$ or Ω, both of which are in $\mathcal{F}_t$ for any $t \geq 0$. Therefore, any constant c in $\mathbb{R}_+$ is a stopping time.

Proof of (2). Assume first that T is a stopping time. Then $(T < t) = \cup_{n=1}^\infty (T \leq t - \frac{1}{n}) \in \mathcal{F}_t$, proving the implication towards the right. Conversely, assume $(T < t) \in \mathcal{F}_t$ for all $t \geq 0$. We then obtain $(T \leq t) = \cap_{k=n}^\infty (T < t + \frac{1}{k})$ for all n. This shows $(T \leq t) \in \mathcal{F}_{t+\frac{1}{n}}$ for all $n \geq 1$. Since $(\mathcal{F}_t)$ is decreasing and n is arbitrary, $(T \leq t) \in \cap_{n=1}^\infty \mathcal{F}_{t+\frac{1}{n}} = \cap_{n=1}^\infty \cap_{s \geq t + \frac{1}{n}} \mathcal{F}_s = \cap_{s>t} \mathcal{F}_s$. By right-continuity of the filtration, $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$, so we conclude $(T \leq t) \in \mathcal{F}_t$, proving that T is a stopping time.

Proof of (3). Assume that S and T are stopping times and let $t \geq 0$. We then have $(S \wedge T \leq t) = (S \leq t) \cup (T \leq t) \in \mathcal{F}_t$, so $S \wedge T$ is a stopping time. Likewise, we obtain $(S \vee T \leq t) = (S \leq t) \cap (T \leq t) \in \mathcal{F}_t$, so $S \vee T$ is a stopping time as well. Finally, consider the sum $S + T$. Let $n \geq 1$ and fix ω. If $S(\omega)$ and $T(\omega)$ are finite, there are $q, q' \in \mathbb{Q}_+$ such that $q \leq S(\omega) \leq q + \frac{1}{n}$ and $q' \leq T(\omega) \leq q' + \frac{1}{n}$. In particular, $q + q' \leq S(\omega) + T(\omega)$ and $S(\omega) + T(\omega) \leq q + q' + \frac{2}{n}$. Next, if $S(\omega) + T(\omega) \leq t$, it holds in particular that both $S(\omega)$ and $T(\omega)$ are finite. Therefore, with $\Theta_t = \{q, q' \in \mathbb{Q}_+ \mid q + q' \leq t\}$, we find

$$(S + T \leq t) = \cap_{n=1}^\infty \cup_{(q,q') \in \Theta_t} (S \leq q + \tfrac{1}{n}) \cap (T \leq q' + \tfrac{1}{n}).$$

Now, the sequence of sets $\cup_{(q,q')\in\Theta_t}(S \leq q+\frac{1}{n})\cap(T \leq q'+\frac{1}{n})$ is decreasing in n, and therefore we have for any $k \geq 1$ that $(S+T \leq t) = \cap_{n=k}^{\infty}\cup_{(q,q')\in\Theta_t}(S \leq q+\frac{1}{n})\cap(T \leq q'+\frac{1}{n}) \in \mathcal{F}_{t+\frac{1}{k}}$. In particular, $(S+T \leq t) \in \mathcal{F}_s$ for any $s > t$, and so, by right-continuity of the filtration, $(S+T \leq t) \in \cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$, proving that $S+T$ is a stopping time.

Proof of (4). Let T be a stopping time and let $F \in \mathcal{F}_T$. Then $(T_F \leq t) = (T \leq t)\cap F \in \mathcal{F}_t$, as was to be proven.

Proof of (5). Let $A \in \mathcal{F}_S$, so that $A \cap (S \leq t) \in \mathcal{F}_t$ for all $t \geq 0$. Since $S \leq T$, we have $(T \leq t) \subseteq (S \leq t)$ and so $A \cap (T \leq t) = A \cap (S \leq t) \cap (T \leq t) \in \mathcal{F}_t$, yielding $A \in \mathcal{F}_T$. □

For the next result, we recall that for any nonempty $A \subseteq \mathbb{R}$, it holds that $\sup A \leq t$ if and only if $s \leq t$ for all $s \in A$, and $\inf A < t$ if and only if there is $s \in A$ such that $s < t$.

Lemma 1.1.10. *Let (T_n) be a sequence of stopping times, then $\sup_n T_n$ and $\inf_n T_n$ are stopping times as well.*

Proof. Assume that T_n is a stopping time for each n. Fix $t \geq 0$, we then then have that $(\sup_n T_n \leq t) = \cap_{n=1}^{\infty}(T_n \leq t) \in \mathcal{F}_t$, so $\sup_n T_n$ is a stopping time. Likewise, using the second statement of Lemma 1.1.9, we find $(\inf_n T_n < t) = \cup_{n=1}^{\infty}(T_n < t) \in \mathcal{F}_t$, so $\inf_n T_n$ is a stopping time. □

Lemma 1.1.11. *Let T and S be stopping times. Assume that Z is $\mathcal{F}_S$ measurable. It then holds that both $Z1_{(S<T)}$ and $Z1_{(S\leq T)}$ are $\mathcal{F}_{S\wedge T}$ measurable.*

Proof. We first show $(S < T) \in \mathcal{F}_{S\wedge T}$. To prove the result, it suffices to show that the set $(S < T) \cap (S \wedge T \leq t)$ is in $\mathcal{F}_t$ for all $t \geq 0$. To this end, we begin by noting that $(S < T)\cap(S\wedge T \leq t) = (S < T)\cap(S \leq t)$. Consider some $\omega \in \Omega$ such that $S(\omega) < T(\omega)$ and $S(\omega) \leq t$. If $t < T(\omega)$, $S(\omega) \leq t < T(\omega)$. If $T(\omega) \leq t$, there is some $q \in \mathbb{Q}\cap[0,t]$ such that $S(\omega) \leq q < T(\omega)$. We thus obtain $(S < T) \cap (S \wedge T \leq t) = \cup_{q\in\mathbb{Q}\cap[0,t]\cup\{t\}}(S \leq q) \cap (q < T)$, which is in $\mathcal{F}_t$, showing $(S < T) \in \mathcal{F}_{S\wedge T}$. We next show that $Z1_{(S<T)}$ is $\mathcal{F}_{S\wedge T}$ measurable. Let $B \in \mathcal{B}$ with B not containing zero. As this type of sets generate $\mathcal{B}$, it will suffice to show that $(Z1_{(S<T)} \in B) \cap (S \wedge T \leq t) \in \mathcal{F}_t$ for all $t \geq 0$. To obtain this, we rewrite

$$\begin{aligned}(Z1_{(S<T)} \in B) \cap (S \wedge T \leq t) &= (Z \in B) \cap (S < T) \cap (S \wedge T \leq t)\\ &= (Z \in B) \cap (S < T) \cap (S \leq t).\end{aligned}$$

Since Z is $\mathcal{F}_S$ measurable, $(Z \in B) \cap (S \leq t) \in \mathcal{F}_t$. And by what we have already shown, $(S < T) \in \mathcal{F}_S$, so $(S < T) \cap (S \leq t) \in \mathcal{F}_t$. Thus, the above is in $\mathcal{F}_t$, as desired.

Finally, we show that $Z1_{(S\leq T)}$ is $\mathcal{F}_{S\wedge T}$ measurable. Let $B\in\mathcal{B}$ with B not containing zero. As above, it suffices to show that for any $t\geq 0$, $(Z1_{(S\leq T)}\in B)\cap(S\wedge T\leq t)\in\mathcal{F}_t$. To obtain this, we first write

$$\begin{aligned}(Z1_{(S\leq T)}\in B)\cap(S\wedge T\leq t) &= (Z\in B)\cap(S\leq T)\cap(S\wedge T\leq t)\\ &= (Z\in B)\cap(S\leq t)\cap(S\leq T)\cap(S\wedge T\leq t).\end{aligned}$$

Since $Z\in\mathcal{F}_S$, we find $(Z\in B)\cap(S\leq t)\in\mathcal{F}_t$. And since we know $(T<S)\in\mathcal{F}_{T\wedge S}$, $(S\leq T)=(T<S)^c\in\mathcal{F}_{S\wedge T}$, so $(S\leq T)\cap(S\wedge T\leq t)\in\mathcal{F}_t$. This demonstrates $(Z1_{(S\leq T)}\in B)\cap(S\wedge T\leq t)\in\mathcal{F}_t$, as desired. □

Lemma 1.1.12. *Let X be progressively measurable, and let T be a stopping time. Then $X_T1_{(T<\infty)}$ is $\mathcal{F}_T$ measurable and X^T is progressively measurable.*

Proof. We first prove that the stopped process X^T is progressively measurable. Fix $t\geq 0$, we need to show that $X^T_{|[0,t]\times\Omega}$ is $\mathcal{B}_t\otimes\mathcal{F}_t$ measurable, which means that we need to show that the mapping from $[0,t]\times\Omega$ to $\mathbb{R}$ given by $(s,\omega)\mapsto X_{T(\omega)\wedge s}(\omega)$ is $\mathcal{B}_t\otimes\mathcal{F}_t$-$\mathcal{B}$ measurable. To this end, note that whenever $0\leq s\leq t$,

$$\{(u,\omega)\in[0,t]\times\Omega\mid T(\omega)\wedge u\leq s\} \quad=\quad ([0,t]\times(T\leq s))\cup([0,s]\times\Omega)\in\mathcal{B}_t\otimes\mathcal{F}_t,$$

so the mapping from $[0,t]\times\Omega$ to $[0,t]$ given by $(s,\omega)\mapsto T(\omega)\wedge s$ is $\mathcal{B}_t\otimes\mathcal{F}_t$-$\mathcal{B}_t$ measurable. And as the mapping from $[0,t]\times\Omega$ to Ω given by $(s,\omega)\mapsto\omega$ is $\mathcal{B}_t\otimes\mathcal{F}_t$-$\mathcal{F}_t$ measurable, we conclude that the mapping from $[0,t]\times\Omega$ to $[0,t]\times\Omega$ given by $(s,\omega)\mapsto(T(\omega)\wedge s,\omega)$ is $\mathcal{B}_t\otimes\mathcal{F}_t$-$\mathcal{B}_t\otimes\mathcal{F}_t$ measurable, since it has measurable coordinates. As X is progressive, the mapping from $[0,t]\times\Omega$ to $\mathbb{R}$ given by $(s,\omega)\mapsto X_s(\omega)$ is $\mathcal{B}_t\otimes\mathcal{F}_t$-$\mathcal{B}$ measurable. Therefore, the composite mapping from $[0,t]\times\Omega$ to $\mathbb{R}$ given by $(s,\omega)\mapsto X_{T(\omega)\wedge s}(\omega)$ is $\mathcal{B}_t\otimes\mathcal{F}_t$-$\mathcal{B}$ measurable. This shows that X^T is progressively measurable.

In order to prove that X_T is $\mathcal{F}_T$ measurable, we note that for any $B\in\mathcal{B}$, we have that $(X_T1_{(T<\infty)}\in B)\cap(T\leq t)=(X^T_t\in B)\cap(T\leq t)$. Now, X^T_t is $\mathcal{F}_t$ measurable since X^T is progressive and therefore adapted by Lemma 1.1.8, and $(T\leq t)\in\mathcal{F}_t$ since T is a stopping time. Thus, $(X_T1_{(T<\infty)}\in B)\cap(T\leq t)\in\mathcal{F}_t$, and we conclude $(X_T1_{(T<\infty)}\in B)\in\mathcal{F}_T$. □

The remaining results of the section give some results on stopping times related to càdlàg adapted stochastic processes and their jumps.

Lemma 1.1.13. *Let X be an adapted càdlàg process, and let U be an open set in $\mathbb{R}$. Define $T=\inf\{t\geq 0\mid X_t\in U\}$. Then T is a stopping time.*

Proof. Note that if $s \geq 0$ and (s_n) is a sequence converging downwards to s, we have by right-continuity that X_{s_n} converges to X_s and so, since U is open, $(X_s \in U) \subseteq \cup_{n=1}^{\infty}(X_{s_n} \in U)$. Using the density properties of $\mathbb{Q}_+$ in relation to $\mathbb{R}_+$, we then find

$$\begin{aligned}(T < t) &= (\ \exists\, s \in \mathbb{R}_+ : s < t \text{ and } X_s \in U) \\ &= (\ \exists\, s \in \mathbb{Q}_+ : s < t \text{ and } X_s \in U) = \cup_{s \in \mathbb{Q}_+, s < t}(X_s \in U),\end{aligned}$$

and since X is adapted, we have $(X_s \in U) \in \mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s < t$, proving that $(T < t) \in \mathcal{F}_t$. By Lemma 1.1.9, this implies that T is a stopping time. □

Lemma 1.1.14. *Let X be a càdlàg adapted process, and let U be an open set in $\mathbb{R}$. Define $T = \inf\{t \geq 0 \mid \Delta X_t \in U\}$. Then T is a stopping time.*

Proof. As X is càdlàg, ΔX is pathwisely zero everywhere except for on a countable set, and so T is identically zero if U contains zero. In this case, it is immediate that T is a stopping time. Thus, it suffices to prove the result in the case where U does not contain zero. Therefore, assume that U is an open set not containing zero. By Lemma 1.1.9, it suffices to show $(T < t) \in \mathcal{F}_t$ for $t > 0$. To this end, fix $t > 0$. As $\Delta X_0 = 0$ and U does not contain zero, we have

$$(T < t) \quad = \quad (\ \exists\, s \in (0, \infty) : s < t \text{ and } X_s - X_{s-} \in U).$$

Let $F_m = \{x \in \mathbb{R} \mid \forall\, y \in U^c : |x - y| \geq 1/m\}$, F_m is an intersection of closed sets and therefore itself closed. Clearly, $(F_m)_{m \geq 1}$ is increasing, and since U is open, $U = \cup_{m=1}^{\infty} F_m$. Also, $F_m \subseteq F_{m+1}^{\circ}$, where F_{m+1}° denotes the interior of F_{m+1}. Let Θ_k be the subset of $\mathbb{Q}^2$ defined by $\Theta_k = \{(p, q) \in \mathbb{Q}^2 \mid 0 < p < q < t, |p - q| \leq \frac{1}{k}\}$. We will prove the result by showing that

$$\begin{aligned}&(\exists s \in (0, \infty) : s < t \text{ and } X_s - X_{s-} \in U) \\ &= \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} \cup_{(p,q) \in \Theta_k} (X_q - X_p \in F_m).\end{aligned}$$

To obtain this, first consider the inclusion towards the right. Assume that there is $0 < s < t$ such that $X_s - X_{s-} \in U$. Take m such that $X_s - X_{s-} \in F_m$. As $F_m \subseteq F_{m+1}^{\circ}$, we then have $X_s - X_{s-} \in F_{m+1}^{\circ}$ as well. As F_{m+1}° is open and as X is càdlàg, it holds that there is $\varepsilon > 0$ such that whenever $p, q \geq 0$ with $p \in (s - \varepsilon, s)$ and $q \in (s, s + \varepsilon)$, $X_q - X_p \in F_{m+1}^{\circ}$. Take $n \in \mathbb{N}$ such that $1/2n < \varepsilon$. We now claim that for $k \geq n$, there is $(p, q) \in \Theta_k$ such that $X_q - X_p \in F_{m+1}$. To prove this, let $k \geq n$ be given. By the density properties of $\mathbb{Q}_+$ in $\mathbb{R}_+$, there are elements $p, q \in \mathbb{Q}$ with $p, q \in (0, t)$ such that $p \in (s - 1/2k, s)$ and $q \in (s, s + 1/2k)$. In particular, then $0 < p < q < t$ and $|p - q| \leq |p - s| + |s - q| \leq 1/k$, so $(p, q) \in \Theta_k$. As

$1/2k \leq 1/2n < \varepsilon$, we have $p \in (s-\varepsilon, s)$ and $q \in (s, s+\varepsilon)$, and so $X_q - X_p \in F^\circ_{m+1} \subseteq F_{m+1}$. This proves the inclusion towards the right.

Now consider the inclusion towards the left. Assume that there is $m \geq 1$ and $n \geq 1$ such that for all $k \geq n$, there exists $(p,q) \in \Theta_k$ with $X_q - X_p \in F_m$. We may use this to obtain sequences $(p_k)_{k\geq n}$ and $(q_k)_{k\geq n}$ with the properties that $p_k, q_k \in \mathbb{Q}$, $0 < p_k < q_k < t$, $|p_k - q_k| \leq \frac{1}{k}$ and $X_{q_k} - X_{p_k} \in F_m$. Putting $p_k = p_n$ and $q_k = q_n$ for $k < n$, we then find that the sequences $(p_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ satisfy $p_k, q_k \in \mathbb{Q}$, $0 < p_k < q_k < t$, $\lim_k |p_k - q_k| = 0$ and $X_{q_k} - X_{p_k} \in F_m$. As all sequences of real numbers contain a monotone subsequence, we may by taking two consecutive subsequences and renaming our sequences obtain the existence of two monotone sequences (p_k) and (q_k) in $\mathbb{Q}$ with $0 < p_k < q_k < t$, $\lim_k |p_k - q_k| = 0$ and $X_{q_k} - X_{p_k} \in F_m$. As bounded monotone sequences are convergent, both (p_k) are (q_k) are then convergent, and as $\lim_k |p_k - q_k| = 0$, the limit $s \geq 0$ is the same for both sequences.

We wish to argue that $s > 0$, that $X_{s-} = \lim_k X_{p_k}$ and that $X_s = \lim_k X_{q_k}$. To this end, recall that U does not contain zero, and so as $F_m \subseteq U$, F_m does not contain zero either. Also note that as both (p_k) and (q_k) are monotone, the limits $\lim_k X_{p_k}$ and $\lim_k X_{q_k}$ exist and are either equal to X_s or X_{s-}. As $X_{q_k} - X_{p_k} \in F_m$ and F_m is closed and does not contain zero, $\lim_k X_{q_k} - \lim_k X_{p_k} = \lim_k X_{q_k} - X_{p_k} \neq 0$. From this, we can immediately conclude that $s > 0$, as if $s = 0$, we would obtain that both $\lim_k X_{q_k}$ and $\lim_k X_{p_k}$ were equal to X_s, yielding $\lim_k X_{q_k} - \lim_k X_{p_k} = 0$, a contradiction. Also, we cannot have that both limits are X_s or that both limits are X_{s-}, and so only two cases are possible, namely that $X_s = \lim_k X_{q_k}$ and $X_{s-} = \lim_k X_{p_k}$ or that $X_s = \lim_k X_{p_k}$ and $X_{s-} = \lim_k X_{q_k}$. We wish to argue that the former holds. If $X_s = X_{s-}$, this is trivially the case. Assume that $X_s \neq X_{s-}$ and that $X_s = \lim_k X_{p_k}$ and $X_{s-} = \lim_k X_{q_k}$. If $q_k \geq s$ from a point onwards or $p_k < s$ from a point onwards, we obtain $X_s = X_{s-}$, a contradiction. Therefore, $q_k < s$ infinitely often and $p_k \geq s$ infinitely often. By monotonicity, $q_k < s$ and $p_k \geq s$ from a point onwards, a contradiction with $p_k < q_k$. We conclude $X_s = \lim_k X_{q_k}$ and $X_{s-} = \lim_k X_{p_k}$, as desired.

In particular, $X_s - X_{s-} = \lim_k X_{q_k} - X_{p_k}$. As $X_{q_k} - X_{p_k} \in F_m$ and F_m is closed, we obtain $X_s - X_{s-} \in F_m \subseteq U$. Next, note that if $s = t$, we have $p_k, q_k < s$ for all k, yielding that both sequences must be increasing and $X_s = \lim X_{q_k} = X_{s-}$, a contradiction with the fact that $X_s - X_{s-} \neq 0$ as $X_s - X_{s-} \in U$. Thus, $0 < s < t$. This proves the existence of $s \in (0, \infty)$ with $s < t$ such that $X_s - X_{s-} \in U$, and so proves the inclusion towards the left.

We have now shown the announced equality. Now, as X_s is $\mathcal{F}_t$ measurable for all $0 \leq s \leq t$, it holds that the set $\cup_{m=1}^\infty \cup_{n=1}^\infty \cap_{k=n}^\infty \cup_{(p,q)\in\Theta_k} (X_q - X_p \in F_m)$ is $\mathcal{F}_t$ measurable as well. We conclude that $(T < t) \in \mathcal{F}_t$ and so T is a stopping time. $\square$

In order to formulate and prove the next lemma, we introduce the notion of stochastic intervals. Let S and T be two stopping times. We then define the subset $]\!]S,T]\!]$ of $\mathbb{R}_+ \times \Omega$ by putting $]\!]S,T]\!] = \{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid S(\omega) < t \leq T(\omega)\}$. We define $[\![S,T]\!]$, $]\!]S,T[\![$ and $[\![S,T[\![$ in analogous manner as subsets of $\mathbb{R}_+ \times \Omega$. Note in particular that even if T is infinite, the sets $]\!]S,T]\!]$ and $[\![S,T]\!]$ do not contain infinity. We also use the notational shorthand $[\![T]\!] = [\![T,T]\!]$ and refer to $[\![T]\!]$ as the graph of the stopping time T.

Lemma 1.1.15. *Let X be a càdlàg adapted process. Define $T_1^k = \inf\{t \geq 0 \mid |\Delta X_t| > \frac{1}{k}\}$ for $k \geq 1$, and recursively for $n \geq 2$, $T_n^k = \inf\{t > T_{n-1}^k \mid |\Delta X_t| > \frac{1}{k}\}$. Then T_n^k is a stopping time for all $k \geq 1$ and $n \geq 1$, $|\Delta X_{T_n^k}| > \frac{1}{k}$ whenever T_n^k is finite and it holds that $\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} = \cup_{k=1}^\infty \cup_{n=1}^\infty [\![T_n^k]\!]$.*

Proof. First note that by Lemma A.2.3, since X has càdlàg paths, it holds that the set $\{t \geq 0 \mid |\Delta X_t| > \frac{1}{k}\}$ always has finite intersection with any bounded interval. From this, we conclude that for all $k \geq 1$ and $n \geq 1$, $|\Delta X_{T_n^k}| > \frac{1}{k}$ whenever T_n^k is finite.

Next, fix $k \geq 1$. We will prove by induction that T_n^k is a stopping time for all $n \geq 1$. The induction start follows from Lemma 1.1.14. Fix $n \geq 2$ and assume that the results have been proven for $1, \ldots, n-1$. Define a process Y^{kn} by putting $Y^{kn} = \sum_{i=1}^{n-1} \Delta X_{T_i^k} 1_{[\![T_i^k,\infty[\![}$. As $\{t \geq 0 \mid |\Delta X_t| > \frac{1}{k}\}$ has finite intersection with any bounded interval, the set $\{T_1^k, \ldots, T_{n-1}^k\}$ contains all jump times of X with absolute value strictly larger than $\frac{1}{k}$ on $[0, T_n^k)$. Therefore, the process $X - Y^{kn}$ has no jumps strictly larger than $\frac{1}{k}$ on $[0, T_n^k)$, and so it holds that $T_n^k = \inf\{t \geq 0 \mid |\Delta(X - Y^{kn})_t| > \frac{1}{k}\}$. Thus, if we can show that $X - Y^{kn}$ is càdlàg adpated, Lemma 1.1.14 will yield that T_n^k is a stopping time.

To this end, note that X is càdlàg adapted, it suffices to show that Y^{kn} is càdlàg adapted. The càdlàg property is immediate, we prove adaptedness. Fixing $t \geq 0$ observe that we have $Y_t^{kn} = \sum_{i=1}^{n-1} \Delta X_{T_i^k} 1_{(T_i^k \leq t)} = \sum_{i=1}^{n-1} \Delta X_{T_i^k} 1_{(T_i^k < \infty)} 1_{(T_i^k \leq t)}$. By our induction assumptions, the variables $T_1^k, \ldots, T_{n-1}^k$ are stopping times. Therefore, for $i \leq n-1$, Lemma 1.1.12 shows that $X_{T_i^k} 1_{(T_i^k < \infty)}$ is $\mathcal{F}_{T_i^k}$ measurable, and so Lemma 1.1.11 shows that $X_{T_i^k} 1_{(T_i^k < \infty)} 1_{(T_i^k \leq t)}$ is $\mathcal{F}_t$ measurable. Thus, Y_t^{kn} is $\mathcal{F}_t$ measurable, proving that Y^{kn} is adapted. Applying Lemma 1.1.14, we conclude that T_n^k is a stopping time, concluding the induction proof.

Finally, again from the fact that the set $\{t \geq 0 \mid |\Delta X_t| > \frac{1}{k}\}$ always has finite intersection with any bounded interval, we have $\{t \geq 0 \mid |\Delta X_t| > \frac{1}{k}\} = \{T_n^k | n \geq 1, T_n^k < \infty\}$, and this shows the identify for the jump set $\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\}$. $\square$

Lemma 1.1.16. *Let X be a càdlàg adapted process. If it holds for all bounded stopping times T that ΔX_T is almost surely zero, then X is almost surely continuous.*

Proof. Assume that ΔX_T is almost surely zero for all bounded stopping times. Let T be any stopping time. On the set $(T < \infty)$, $T = T \wedge n$ for n large enough, depending on ω. Therefore, $\Delta X_T = \Delta X_T 1_{(T<\infty)} = \lim_n \Delta X_{T\wedge n} 1_{(T<\infty)}$, and so ΔX_T is almost surely zero for all stopping times T. By Lemma 1.1.15, there exists a countable family of stopping times (T_n) such that $\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} = \cup_{n=1}^\infty [\![T_n]\!]$, yielding $\Delta X = \sum_{n=1}^\infty \Delta X_{T_n} 1_{[\![T_n]\!]}$. By what we already have shown, this is almost surely zero, and so X is almost surely continuous. This concludes the proof. □

The following lemma shows that all continuous adapted processes are bounded in a local sense. That this is not the case for càdlàg adapted processes is part of what makes the theory of general martingales more difficult than the theory of continuous martingales.

Lemma 1.1.17. *Let X be any continuous adapted process with initial value zero. Defining $T_n = \inf\{t \geq 0 \mid |X_t| > n\}$, (T_n) is a sequence of stopping times increasing pointwise to infinity, and the process X^{T_n} is bounded by n.*

Proof. By Lemma 1.1.13, (T_n) is a sequence of stopping times. We prove that X^{T_n} is bounded by n. If T_n is infinite, $X_t \leq n$ for all $t \geq 0$, so on $(T_n = \infty)$, X^{T_n} is bounded by n. If T_n is finite, note that for all $\varepsilon > 0$, there is $t \geq 0$ with $T_n \leq t < T_n + \varepsilon$ such that $|X_t| > n$. Therefore, by continuity, $|X_{T_n}| \geq n$. In particular, as X has initial value zero, T_n cannot take the value zero. Therefore, there is $t \geq 0$ with $t < T_n$. For all such t, $|X_t| \leq n$. Therefore, again by continuity, $|X_{T_n}| \leq n$, and we conclude that in this case as well, X^{T_n} is bounded by n. Note that we have also shown that $|X_{T_n}| = n$ whenever T_n is finite.

It remains to show that T_n converges pointwise to infinity. To obtain this, note that as X is continuous, X is bounded on compacts. If for some samle path we have that $T_n \leq a$ for all n, we would have $|X_{T_n}| = n$ for all n and so X would be unbounded on $[0,a]$. This is a contradiction, since X has continuous sample paths and therefore is bounded on compact sets. Therefore, (T_n) is unbounded for every sample path. As T_n is increasing, this shows that T_n converges to infinity pointwise. □

1.2 Continuous-time martingales

In this section, we consider càdlàg martingales in continuous time. We say that a process M is a continuous-time martingale if M is adapted and for any $0 \le s \le t$, $E(M_t|\mathcal{F}_s) = M_s$ almost surely. In the same manner, if M is adapted and for any $0 \le s \le t$, we have $E(M_t|\mathcal{F}_s) \le M_s$ almost surely, we say that M is a supermartingale, and if M is adapted and for any $0 \le s \le t$, $E(M_t|\mathcal{F}_s) \ge M_s$ almost surely, we say that M is a submartingale. We are interested in transferring the results known from discrete-time martingales to the continuous-time setting, mainly the criteria for almost sure convergence, $\mathcal{L}^1$ convergence and the optional sampling theorem. The classical results from discrete-time martingale theory are reviewed in Appendix A.4. We will only take interest in martingales whose sample paths are càdlàg. This is not a significant restriction, as we have assumed that our filtered probability space satisfies the usual conditions, so all martingales will have a càdlàg version, see for example Theorem II.67.7 of Rogers & Williams (2000a).

We will for the most part only take interest in martingales M whose initial value is zero, meaning that $M_0 = 0$, in order to simplify the exposition. We denote the space of martingales in continuous time with initial value zero by $\mathcal{M}$. By $\mathcal{M}^u$, we denote the elements of $\mathcal{M}$ which are uniformly integrable, and by $\mathcal{M}^b$, we denote the elements of $\mathcal{M}^b$ which are bounded in the sense that there exists $c > 0$ such that $|M_t| \le c$ for all $t \ge 0$. Clearly, $\mathcal{M}$ and $\mathcal{M}^b$ are both vector spaces, and by Lemma A.3.4, $\mathcal{M}^u$ is a vector space as well. Subspaces of continuous martingales are denoted by adding a **c** to the corresponding spaces, such that $\mathbf{c}\mathcal{M}$ denotes the subspace of elements of $\mathcal{M}$ with continuous paths, $\mathbf{c}\mathcal{M}^u$ is the space of continuous processes in $\mathcal{M}^u$ and $\mathbf{c}\mathcal{M}^b$ is space of the continuous processes in $\mathcal{M}^b$.

The main lemma for transferring the results of discrete-time martingale theory to continuous-time martingale theory is the following.

Lemma 1.2.1. *Let M be a continuous-time martingale, supermartingale or submartingale, and let (t_n) be an increasing sequence in $\mathbb{R}_+$. Then $(\mathcal{F}_{t_n})_{n\ge1}$ is a discrete-time filtration, and the process $(M_{t_n})_{n\ge1}$ is a discrete-time martingale, supermartingale or submartingale, respectively, with respect to the filtration $(\mathcal{F}_{t_n})_{n\ge1}$.*

Proof. This follows immediately from the definition of continuous-time and discrete-time martingales, supermartingales and submartingales. □

Lemma 1.2.2 (Doob's upcrossing lemma)**.** *Let Z be a càdlàg supermartingale bounded in $\mathcal{L}^1$. Define $U(Z,a,b) = \sup\{m \mid \exists\, 0 \le s_1 < t_1 < \cdots s_m < t_m : Z_{s_k} < a, Z_{t_k} > b, k \le m\}$ for* *any $a, b \in \mathbb{R}$ with $a < b$. We refer to $U(Z, a, b)$ as the number of upcrossings from a to b by M. Then $U(Z, a, b)$ is $\mathcal{F}$ measurable and it holds that*

$$EU(Z, a, b) \leq \frac{|a| + \sup_t E|Z_t|}{b - a}.$$

Proof. We will prove the result by reducing to the case of upcrossings relative to a countable number of timepoints and applying Lemma 1.2.1 and the discrete-time upcrossing result of Lemma A.4.1. For any $D \subseteq \mathbb{R}$, we define

$$U(Z, a, b, D) = \sup\{m \mid \exists\, 0 \leq s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in D, Z_{s_i} < a, Z_{t_i} > b, i \leq m\},$$

and we refer to $U(Z, a, b, D)$ as the number of upcrossings from a to b at the timepoints in D. Define $\mathbb{D}_+ = \{k2^{-n} \mid k \geq 0, n \geq 1\}$, we refer to $\mathbb{D}_+$ as the dyadic nonnegative rationals. It holds that $\mathbb{D}_+$ is dense in $\mathbb{R}_+$. Now, as Z is right-continuous, we find that for any finite sequence $0 \leq s_1 < t_1 < \cdots s_m < t_m$ such that $s_i, t_i \in \mathbb{R}_+$ with $Z_{s_i} < a$ and $Z_{t_i} > b$ for $i \leq m$, there exists $0 \leq p_1 < q_1 < \cdots p_m < q_m$ such that $p_i, q_i \in \mathbb{D}_+$ with $Z_{p_i} < a$ and $Z_{q_i} > b$ for $i \leq m$. Therefore, $U(Z, a, b) = U(Z, a, b, \mathbb{D}_+)$. In other words, it suffices to consider upcrossings at dyadic nonnegative rational timepoints. In order to use this to prove that $U(Z, a, b)$ is $\mathcal{F}$ measurable, note that for any $m \geq 1$, we have

$$\begin{aligned} & (\,\exists\, 0 \leq s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in \mathbb{D}_+, Z_{s_i} < a, Z_{t_i} > b, i \leq m) \\ = \; & \cup\{(Z_{s_i} < a, Z_{t_i} > b \text{ for all } i \leq m) \mid 0 \leq s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in \mathbb{D}_+\}, \end{aligned}$$

which is in $\mathcal{F}$, as $(Z_{s_i} < a, Z_{t_i} > b$ for all $i \leq m)$ is $\mathcal{F}$ measurable, and all subsets of $\cup_{n=1}^\infty \mathbb{D}_+^n$ are countable. Here, $\mathbb{D}_+^n$ denotes the n-fold product of $\mathbb{D}_+$. From these observations, we conclude that the set $(\,\exists\, 0 \leq s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in \mathbb{D}_+, Z_{s_i} < a, Z_{t_i} > b, i \leq m)$ is $\mathcal{F}$ measurable. Denote this set by A_m, we then have $U(Z, a, b)(\omega) = \sup\{m \in \mathbb{N} \mid \omega \in A_m\}$, so that in particular $(U(Z, a, b) \leq m) = \cap_{k=m+1}^\infty A_k^c \in \mathcal{F}$ and so $U(Z, a, b)$ is $\mathcal{F}$ measurable.

It remains to prove the bound for the mean of $U(Z, a, b)$. Putting $t_k^n = k2^{-n}$ and defining $D_n = \{t_k^n \mid k \geq 0\}$, we obtain $\mathbb{D}_+ = \cup_{n=1}^\infty D_n$. We then have

$$\begin{aligned} & \sup\{m \mid \exists\, 0 \leq s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in \mathbb{D}_+, Z_{s_i} < a, Z_{t_i} > b, i \leq m\} \\ = \; & \sup \cup_{n=1}^\infty \{m \mid \exists\, 0 \leq s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in D_n, Z_{s_i} < a, Z_{t_i} > b, i \leq m\} \\ = \; & \sup_n \sup\{m \mid \exists\, 0 \leq s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in D_n, Z_{s_i} < a, Z_{t_i} > b, i \leq m\}, \end{aligned}$$

so $U(Z, a, b, \mathbb{D}_+) = \sup_n U(Z, a, b, D_n)$. Now fix $n \in \mathbb{N}$. As $(t_k^n)_{k \geq 0}$ is an increasing sequence, Lemma 1.2.1 shows that $(Z_{t_k^n})_{k \geq 0}$ is a discrete-time supermartingale with respect to the filtration $(\mathcal{F}_{t_k^n})_{k\geq 0}$. As $(Z_t)_{t\geq 0}$ is bounded in $\mathcal{L}^1$, so is $(Z_{t_k^n})_{k\geq 0}$. Therefore, Lemma A.4.1 yields

$$EU(Z,a,b,D_n) \leq \frac{|a| + \sup_k E|Z_{t_k^n}|}{b-a} \leq \frac{|a| + \sup_t E|Z_t|}{b-a}.$$

As (D_n) is increasing, $U(Z,a,b,D_n)$ is increasing, so the monotone convergence theorem and our previous results yield

$$\begin{aligned} EU(Z,a,b) &= EU(Z,a,b,\mathbb{D}_+) = E\sup_n U(Z,a,b,D_n) \\ &= E\lim_n U(Z,a,b,D_n) = \lim_n EU(Z,a,b,D_n) \leq \frac{|a| + \sup_t E|Z_t|}{b-a}. \end{aligned}$$

This concludes the proof of the lemma. □

Theorem 1.2.3 (Doob's supermartingale convergence theorem). *Let Z be a càdlàg supermartingale. If Z is bounded in $\mathcal{L}^1$, then Z is almost surely convergent to an integrable limit. If Z is uniformly integrable, then Z also converges in $\mathcal{L}^1$, and the limit Z_∞ satisfies that for all $t \geq 0$, $E(Z_\infty|\mathcal{F}_t) \leq Z_t$ almost surely. If Z is a martingale, the inequality may be exchanged with an equality.*

Proof. Assume that Z is bounded in $\mathcal{L}^1$. Fix $a, b \in \mathbb{Q}$ with $a < b$. By Lemma 1.2.2, the number of upcrossings from a to b made by Z has finite expectation, in particular it is almost surely finite. As $\mathbb{Q}$ is countable, we conclude that it almost surely holds that the number of upcrossings from a to b made by Z is finite for any $a, b \in \mathbb{Q}$. Therefore, Lemma A.2.19 shows that Z is almost surely convergent to a limit in $[-\infty, \infty]$. Using Fatou's lemma, we obtain $E|Z_\infty| = E\liminf_t |Z_t| \leq \liminf_t E|Z_t| \leq \sup_{t\geq 0} E|Z_t|$, which is finite, so we conclude that the limit Z_∞ is integrable.

Assume next that Z is uniformly integrable. In particular, Z is bounded in $\mathcal{L}^1$, so Z_t converges almost surely to some variable Z_∞. Then Z_t also converges in probability, so Lemma A.3.5 shows that Z_t converges to Z_∞ in $\mathcal{L}^1$. We then find that for any $t \geq 0$ that, using Jensen's inequality, $E|E(Z_\infty|\mathcal{F}_t) - E(Z_s|\mathcal{F}_t)| \leq E|Z_\infty - Z_s|$, so $E(Z_s|\mathcal{F}_t)$ tends to $E(Z_\infty|\mathcal{F}_t)$ in $\mathcal{L}^1$ as s tends to infinity, and we get $E(Z_\infty|\mathcal{F}_t) = \lim_{s\to\infty} E(Z_s|\mathcal{F}_t) \leq Z_t$. This proves the results on supermartingales.

In order to obtain the results for the martingale case, next assume that Z is a càdlàg submartingale bounded in $\mathcal{L}^1$. Then $-Z$ is a continuous supermartingale bounded in $\mathcal{L}^1$. From what we already have proved, $-Z$ is almost surely convergent to a finite limit, yielding that Z is almost surely convergent to a finite limit. If Z is uniformly integrable, so is $-Z$, and so we obtain convergence in $\mathcal{L}^1$ as well for $-Z$ and therefore also for Z. Also, we have

$E(-Z_\infty|\mathcal{F}_t) \leq -Z_t$, so $E(Z_\infty|\mathcal{F}_t) \geq Z_t$. Finally, assume that Z is a càdlàg martingale. Then Z is both a càdlàg supermartingale and a càdlàg submartingale, and the result follows. □

Theorem 1.2.4 (Uniformly integrable martingale convergence theorem). *Let $M \in \mathcal{M}$. The following are equivalent:*

(1). M is uniformly integrable.

(2). M is convergent almost surely and in $\mathcal{L}^1$.

(3). There is some integrable variable ξ such that $M_t = E(\xi|\mathcal{F}_t)$ almost surely for $t \geq 0$.

In the affirmative, with M_∞ denoting the limit of M_t almost surely and in $\mathcal{L}^1$, we have for all $t \geq 0$ that $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely, and $M_\infty = E(\xi|\mathcal{F}_\infty)$, where $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$.

Proof. We show that (1) implies (2), that (2) implies (3) and that (3) implies (1).

Proof that (1) **implies** (2)**.** Assume that M is uniformly integrable. By Lemma A.3.3, M is bounded in $\mathcal{L}^1$, and Theorem 1.2.3 shows that M converges almost surely and in $\mathcal{L}^1$.

Proof that (2) **implies** (3)**.** Assume now that M is convergent almost surely and in $\mathcal{L}^1$. Let M_∞ be the limit. Fix $F \in \mathcal{F}_s$ for some $s \geq 0$. As M_t converges to M_∞ in $\mathcal{L}^1$, $1_F M_t$ converges to $1_F M_\infty$ in $\mathcal{L}^1$ as well, and we then obtain

$$E1_F M_\infty = \lim_{t\to\infty} E1_F M_t = \lim_{t\to\infty} E1_F E(M_t|\mathcal{F}_s) = E1_F M_s,$$

proving that $E(M_\infty|\mathcal{F}_s) = M_s$ almost surely for any $s \geq 0$.

Proof that (3) **implies** (1)**.** Finally, assume that there is some integrable variable ξ such that $M_t = E(\xi|\mathcal{F}_t)$. By Lemma A.3.6, M is uniformly integrable.

It remains to prove that in the affirmative, with M_∞ denoting the limit, it holds that for all $t \geq 0$, $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely, and $M_\infty = E(\xi|\mathcal{F}_\infty)$. By what was already shown, in the affirmative case, $M_t = E(M_\infty|\mathcal{F}_t)$. We thus have $E(M_\infty|\mathcal{F}_t) = E(\xi|\mathcal{F}_t)$ almost surely for all $t \geq 0$. In particular, for any $F \in \cup_{t\geq 0}\mathcal{F}_t$, we have $EM_\infty 1_F = EE(\xi|\mathcal{F}_\infty)1_F$. Now let $\mathcal{H} = \{F \in \mathcal{F} | EM_\infty 1_F = EE(\xi|\mathcal{F}_\infty)1_F\}$. We then have that $\mathcal{H}$ is a Dynkin class containing $\cup_{t\geq 0}\mathcal{F}_t$, and $\cup_{t\geq 0}\mathcal{F}_t$ is a generating class for $\mathcal{F}_\infty$, stable under intersections. Therefore, Lemma A.1.1 shows that $\mathcal{F}_\infty \subseteq \mathcal{H}$, so that $EM_\infty 1_F = EE(\xi|\mathcal{F}_\infty)1_F$ for all $F \in \mathcal{F}_\infty$. Since

M_∞ is $\mathcal{F}_\infty$ measurable as the almost sure limit of $\mathcal{F}_\infty$ measurable variables, this implies $M_\infty = E(\xi|\mathcal{F}_\infty)$ almost surely, proving the result. $\square$

Lemma 1.2.5. *If Z is a càdlàg martingale, supermartingale or submartingale, and $c \geq 0$, then the stopped process Z^c is also a càdlàg martingale, supermartingale or submartingale, respectively. Z^c is always convergent almost surely and in $\mathcal{L}^1$ to Z_c. In the martingale case, Z^c is a uniformly integrable martingale.*

Proof. Fix $c \geq 0$. It holds that Z^c is adapted and càdlàg. Let $0 \leq s \leq t$ and consider the supermartingale case. If $c \leq s$, we also have $c \leq t$ and the adaptedness of Z allows us to conclude that

$$E(Z^c_t|\mathcal{F}_s) = E(Z_{t\wedge c}|\mathcal{F}_s) = E(Z_c|\mathcal{F}_s) = Z_c = Z^c_s,$$

and if instead $c \geq s$, the supermartingale property yields

$$E(Z^c_t|\mathcal{F}_s) = E(Z_{t\wedge c}|\mathcal{F}_{s\wedge c}) \leq Z_{s\wedge c} = Z^c_s.$$

This shows that Z^c is a supermartingale. From this, it follows that the submartingale and martingale properties are preserved by stopping at c as well. Also, as Z^c is constant from a deterministic point onwards, Z^c converges almost surely and in $\mathcal{L}^1$ to Z_c. If Z is a martingale, Theorem 1.2.4 shows that Z^c is uniformly integrable. $\square$

Theorem 1.2.6 (Optional sampling theorem)**.** *Let Z be a càdlàg supermartingale, and let S and T be two stopping times with $S \leq T$. If Z is uniformly integrable, then Z is almost surely convergent, Z_S and Z_T are integrable, and $E(Z_T|\mathcal{F}_S) \leq Z_S$. If Z is nonnegative, then Z is almost surely convergent as well and $E(Z_T|\mathcal{F}_S) \leq Z_S$. If instead S and T are bounded, $E(Z_T|\mathcal{F}_S) \leq Z_S$ holds as well, where Z_S and Z_T are integrable. Finally, if Z is a martingale in the uniformly integrable case or the case of bounded stopping times, the inequality may be exchanged with an equality.*

Proof. Assume that Z is a càdlàg supermartingale which is convergent almost surely and in $\mathcal{L}^1$, and let $S \leq T$ be two stopping times. We will prove $E(Z_T|\mathcal{F}_S) \leq Z_S$ in this case and obtain the other cases from this. First, define a mapping S_n by putting $S_n = \infty$ whenever $S = \infty$, and $S_n = k2^{-n}$ when $(k-1)2^{-n} \leq S < k2^{-n}$. We then find

$$\begin{aligned}(S_n \leq t) &= \cup_{k=0}^\infty (S_n = k2^{-n}) \cap (k2^{-n} \leq t) \\ &= \cup_{k=0}^\infty ((k-1)2^{-n} \leq S < k2^{-n}) \cap (k2^{-n} \leq t),\end{aligned}$$

which is in $\mathcal{F}_t$, as $((k-1)2^{-n} \leq S < k2^{-n})$ is in $\mathcal{F}_t$ when $k2^{-n} \leq t$. Therefore, S_n is a stopping time. Furthermore, we have $S \leq S_n$ with S_n converging downwards to S, in the sense that

S_n is decreasing and converges to S. We define (T_n) analogously, such that (T_n) is a sequence of stopping times converging downwards to T, and $(T_n = k2^{-n}) = ((k-1)2^{-n} \leq T < k2^{-n})$. We then obtain that $(T_n = k2^{-n}) = ((k-1)2^{-n} \leq T < k2^{-n}) \subseteq (S < k2^{-n}) = (S_n \leq k2^{-n})$, from which we conclude $S_n \leq T_n$.

We would like to apply the discrete-time optional sampling theorem to the stopping times S_n and T_n. To this end, first note that with $t_k^n = k2^{-n}$, we obtain that by by Lemma 1.2.1, $(Z_{t_k^n})_{k\geq 0}$ is a discrete-time supermartingale with respect to the filtration $(\mathcal{F}_{t_k^n})_{k\geq 0}$. As Z is convergent almost surely and in $\mathcal{L}^1$, so is $(Z_{t_k^n})_{k\geq 0}$, and then Lemma A.3.5 shows that $(Z_{t_k^n})_{k\geq 0}$ is uniformly integrable. Therefore, $(Z_{t_k^n})_{k\geq 0}$ satisfies the requirements in Theorem A.4.5. Furthermore, it holds that $Z_{t_k^n}$ converges to Z_∞. Putting $K_n = S_n 2^n$, K_n takes its values in $\mathbb{N} \cup \{\infty\}$ and $(K_n \leq k) = (S_n \leq k2^{-n}) \in \mathcal{F}_{t_k^n}$, so K_n is a discrete-time stopping time with respect to $(\mathcal{F}_{t_k^n})_{k\geq 0}$. As regards the discrete-time stopping time σ-algebra, we have

$$
\begin{aligned}
\mathcal{F}_{t_{K_n}^n} &= \{F \in \mathcal{F} \mid F \cap (K_n \leq k) \in \mathcal{F}_{t_k^n} \text{ for all } k \geq 0\} \\
&= \{F \in \mathcal{F} \mid F \cap (S_n \leq t_k^n) \in \mathcal{F}_{t_k^n} \text{ for all } k \geq 0\} \\
&= \{F \in \mathcal{F} \mid F \cap (S_n \leq t) \in \mathcal{F}_t \text{ for all } t \geq 0\} = \mathcal{F}_{S_n},
\end{aligned}
$$

where $\mathcal{F}_{t_{K_n}^n}$ denotes the stopping time σ-algebra of the discrete filtration $(\mathcal{F}_{t_k^n})_{k\geq 0}$. Putting $L_n = T_n 2^n$, we find analogous results for the sequence (L_n). Also, since $S_n \leq T_n$, we obtain $K_n \leq L_n$. Therefore, we may now apply Theorem A.4.5 with the uniformly integrable discrete-time supermartingale $(Z_{t_k^n})_{k\geq 0}$ to conclude that Z_{S_n} and Z_{T_n} are integrable and that $E(Z_{T_n}|\mathcal{F}_{S_n}) = E(Z_{t_{L_n}^n}|\mathcal{F}_{t_{K_n}^n}) \leq Z_{t_{K_n}^n} = Z_{S_n}$.

Next, we show that Z_{T_n} converges almost surely and in $\mathcal{L}^1$ to Z_T. This will in particular show that Z_T is integrable. As before, $(Z_{t_k^{n+1}})_{k\geq 0}$ is a discrete-time supermartingale satisfying the requirements in Theorem A.4.5. Also, $(2L_n \leq k) = (2T_n 2^n \leq k) = (T_n \leq k2^{-(n+1)})$, which is in $\mathcal{F}_{t_k^{n+1}}$, so $2L_n$ is a discrete-time stopping time with respect to $(\mathcal{F}_{t_k^{n+1}})_{k\geq 0}$, and it holds that $L_{n+1} = T_{n+1}2^{n+1} \leq T_n 2^{n+1} = 2L_n$. Therefore, applying Theorem A.4.5 to the stopping times $2L_n$ and L_{n+1}, $E(Z_{T_n}|\mathcal{F}_{T_{n+1}}) = E(Z_{t_{2L_n}^{n+1}}|\mathcal{F}_{t_{L_{n+1}}^{n+1}}) \leq Z_{t_{L_{n+1}}^{n+1}} = Z_{T_{n+1}}$. Iterating this relationship, we find that for $n \geq k$, $Z_{T_n} \geq E(Z_{T_k}|\mathcal{F}_{T_n})$. Thus, (Z_{T_n}) is a backwards submartingale with respect to $(\mathcal{F}_{T_n})_{n\geq 0}$. Therefore, $(-Z_{T_n})$ is a backwards supermartingale. Furthermore, as $Z_{T_n} \geq E(Z_{T_k}|\mathcal{F}_{T_n})$ for $n \geq k$, we have $EZ_{T_n} \geq EZ_{T_1}$, so $E(-Z_{T_n}) \leq E(-Z_{T_1})$. This shows that, $\sup_{n\geq 1} E(-Z_{T_n})$ is finite, and so we may apply Theorem A.4.6 to conclude that $(-Z_{T_n})$, and therefore (Z_{T_n}), converges almost surely and in $\mathcal{L}^1$. By right-continuity, we know that Z_{T_n} also converges almost surely to Z_T. By uniqueness of limits, the convergence is in $\mathcal{L}^1$ as well, which in particular implies that Z_T is integrable. Analogously, Z_{S_n} converges to Z_S almost surely and in $\mathcal{L}^1$.

Now fix $F \in \mathcal{F}_S$. As $S \leq S_n$, we have $\mathcal{F}_S \subseteq \mathcal{F}_{S_n}$. Using the convergence of Z_{T_n} to Z_T and Z_{S_n} to Z_S in $\mathcal{L}^1$, we find that $1_F Z_{T_n}$ converges to $1_F Z_T$ and $1_F Z_{S_n}$ converges to $1_F Z_S$ in $\mathcal{L}^1$, so that $E1_F Z_T = \lim_n E1_F Z_{T_n} = \lim_n E1_F E(Z_{T_n}|\mathcal{F}_{S_n}) \leq \lim_n E1_F Z_{S_n} = E1_F Z_S$, and therefore, we conclude $E(Z_T|\mathcal{F}_S) \leq Z_S$, as desired.

This proves that the optional sampling result holds in the case where Z is a supermartingale which is convergent almost surely and in $\mathcal{L}^1$ and $S \leq T$ are two stopping times. We will now obtain the remaining cases from this case.

If Z is a uniformly integrable supermartingale, it is in particular convergent almost surely and in $\mathcal{L}^1$, so we find that the result holds in this case as well. Next, consider the case where we merely assume that Z is a supermartingale and that $S \leq T$ are bounded stopping times. Letting $c \geq 0$ be a bound for S and T, Lemma 1.2.5 shows that Z^c is a supermartingale, and it is clearly convergent almost surely and in $\mathcal{L}^1$. Therefore, as $Z_T = Z^c_T$, we find that Z_T is integrable and that $E(Z_T|\mathcal{F}_S) = E(Z^c_T|\mathcal{F}_S) \leq Z^c_S = Z_S$, proving the result in this case.

Finally, consider the case where Z is nonnegative and $S \leq T$ are any two stopping times. We then find that $E|Z_t| = EZ_t \leq EZ_0$, so Z is bounded in $\mathcal{L}^1$. Therefore, Theorem 1.2.3 shows that Z is almost surely convergent and so Z_T is well-defined. From what we already have shown, $Z_{T\wedge n}$ is integrable and $E(Z_{T\wedge n}|\mathcal{F}_{S\wedge n}) \leq Z_{S\wedge n}$. For any $F \in \mathcal{F}_S$, we find $F \cap (S \leq n) \in \mathcal{F}_{S\wedge n}$ for any n by Lemma 1.1.11. Therefore, we obtain

$$\begin{aligned} E1_F Z_{T\wedge n} &= E1_F 1_{(S\leq n)} Z_{T\wedge n} + E1_F 1_{(S>n)} Z_{T\wedge n} \\ &\leq E1_F 1_{(S\leq n)} Z_{S\wedge n} + E1_F 1_{(S>n)} Z_{S\wedge n} = E1_F Z_{S\wedge n}, \end{aligned}$$

and so, by Lemma A.1.19, $E(Z_{T\wedge n}|\mathcal{F}_S) \leq Z_{S\wedge n}$. Applying Fatou's lemma for conditional expectations, we obtain

$$E(Z_T|\mathcal{F}_S) = E(\liminf_n Z_{T\wedge n}|\mathcal{F}_S) \leq \liminf_n E(Z_{T\wedge n}|\mathcal{F}_S) \leq \liminf_n Z_{S\wedge n} = Z_S,$$

as was to be shown. We have now proved all of the supermartingale statements in the theorem. The martingale results follow immediately from the fact that a martingale is both a supermartingale and a submartingale. □

Lemma 1.2.7. *Let T be a stopping time. If Z is a càdlàg supermartingale, then Z^T is a càdlàg supermartingale as well. In particular, if $M \in \mathcal{M}$, then $M^T \in \mathcal{M}$ as well, and if $M \in \mathcal{M}^u$, then $M^T \in \mathcal{M}^u$ as well.*

Proof. Let a càdlàg supermartingale Z be given, and let T be some stopping time. Fix two timepoints $0 \leq s \leq t$, we need to prove $E(Z^T_t|\mathcal{F}_s) \leq Z^T_s$ almost surely, and to this end, it suffices to show that $E1_F Z^T_t \leq E1_F Z^T_s$ for any $F \in \mathcal{F}_s$. Let $F \in \mathcal{F}_s$ be given. By Lemma 1.1.11, $F \cap (s \leq T)$ is $\mathcal{F}_{s\wedge T}$ measurable, and so Theorem 1.2.6 applied with the two bounded stopping times $T \wedge s$ and $T \wedge t$ yields

$$\begin{aligned} E1_F Z^T_t &= E1_{F\cap(s\leq T)} Z_{T\wedge t} + E1_{F\cap(s>T)} Z_{T\wedge t} \\ &\leq E1_{F\cap(s\leq T)} Z_{T\wedge s} + E1_{F\cap(s>T)} Z_{T\wedge t} \\ &= E1_{F\cap(s\leq T)} Z_{T\wedge s} + E1_{F\cap(s>T)} Z_{T\wedge s} \\ &= E1_F Z^T_s. \end{aligned}$$

Thus, $E(Z^T_t|\mathcal{F}_s) \leq Z^T_s$ and so Z^T is a supermartingale. From this it follows in particular that if $M \in \mathcal{M}$, it holds that $M^T \in \mathcal{M}$ as well. And if $M \in \mathcal{M}^u$, we find that $M^T \in \mathcal{M}$ from what was already shown. Then, by Theorem 1.2.4, $M^T_t = E(M_\infty|\mathcal{F}_{T\wedge t})$, so by Lemma A.3.6, M^T is uniformly integrable, and so $M^T \in \mathcal{M}^u$. □

Next, we prove two extraordinarily useful results, first a criterion for determining when a process is a martingale or a uniformly integrable martingale, and secondly a result showing that a particular class of martingales are evanescent.

Lemma 1.2.8 (Komatsu's lemma)**.** *Let M be a càdlàg adapted process with initial value zero. It holds that $M \in \mathcal{M}$ if and only if M_T is integrable with $EM_T = 0$ for any bounded stopping time T. If the limit $\lim_{t\to\infty} M_t$ exists almost surely, it holds that $M \in \mathcal{M}^u$ if and only if M_T is integrable with $EM_T = 0$ for any stopping time T.*

Proof. We first consider the case where we assume that the limit $\lim_{t\to\infty} M_t$ exists almost surely. By Theorem 1.2.6, we have that if $M \in \mathcal{M}^u$, M_T is integrable and $EM_T = 0$ for any for any stopping time T. Conversely, assume that M_T is integrable and $EM_T = 0$ for any for any stopping time T. We will prove that $M_t = E(M_\infty|\mathcal{F}_t)$ for any $t \geq 0$. To this end, let $F \in \mathcal{F}_t$ and note that by Lemma 1.1.9, t_F is a stopping time, where $t_F = t1_F + \infty 1_{F^c}$, taking only the values t and infinity. We obtain $EM_{t_F} = E1_F M_t + E1_{F^c} M_\infty$, and we also have $EM_\infty = E1_F M_\infty + E1_{F^c} M_\infty$. By our assumptions, both of these are zero, and so $E1_F M_t = E1_F M_\infty$. As M_t is $\mathcal{F}_t$ measurable by assumption, this proves $M_t = E(M_\infty|\mathcal{F}_t)$. From this, we see that M is in $\mathcal{M}$, and by Theorem 1.2.4, M is in $\mathcal{M}^u$.

Consider next the case where we merely assume that M is a càdlàg adapted process with initial value zero. If $M \in \mathcal{M}$, Theorem 1.2.6 shows that M_T is integrable with $EM_T = 0$ for any bounded stopping time. Assume instead that M_T is integrable and $EM_T = 0$ for any bounded stopping time T. From what we already have shown, we then find that M^t is in $\mathcal{M}^u$ for any $t \geq 0$ and therefore, $M \in \mathcal{M}$. □

For the next result, we say that a process X is of finite variation if it has sample paths which are functions of finite variation, see Appendix A.2 for a review of the properties of functions of finite variation. If the process X has finite variation, we denote the variation over $[0,t]$ by $(V_X)_t$, such that $(V_X)_t = \sup \sum_{k=1}^n |X_{t_k} - X_{t_{k-1}}|$, where the supremum is taken over partitions $0 = t_0 < \cdots < t_n = t$ of $[0,t]$.

Lemma 1.2.9. *Let X be càdlàg adapted with finite variation. Then the variation process V_X is càdlàg adapted as well.*

Proof. By Lemma A.2.8, V_X is càdlàg. As for proving that V_X is adapted, note that from Lemma A.2.15, we have $(V_X)_t = \sup \sum_{k=1}^n |X_{q_k} - X_{q_{k-1}}|$, where the supremum is taken over partitions of $[0,t]$ with elements in in $\mathbb{Q}_+ \cup \{t\}$. As $\cup_{n=1}^\infty (\mathbb{Q}_+ \cup \{t\})^n$ is countable, there are only countably many such partitions, and so we find that $(V_X)_t$ is $\mathcal{F}_t$ measurable, since X_q is $\mathcal{F}_t$ measurable whenever $q \le t$. Therefore, V_X is adapted. □

Lemma 1.2.10. *Let X be càdlàg adapted with finite variation. Then $(V_X)^T = V_{X^T}$.*

Proof. Fix $\omega \in \Omega$. With the supremum being over all partitions of $[0, T(\omega) \wedge t]$, we have

$$\begin{aligned}(V_X)^T_t(\omega) &= (V_X)_{T(\omega)\wedge t}(\omega) = \sup \sum_{k=1}^n |X_{t_k}(\omega) - X_{t_{k-1}}(\omega)| \\ &= \sup \sum_{k=1}^n |X^T_{t_k}(\omega) - X^T_{t_{k-1}}(\omega)| = (V_{X^T})_{t\wedge T(\omega)}(\omega) \le (V_{X^T})_t(\omega).\end{aligned}$$

Conversely, with the supremum being over all partitions of $[0,t]$, we also have

$$\begin{aligned}(V_{X^T})_t(\omega) &= \sup \sum_{k=1}^n |X^T_{t_k}(\omega) - X^T_{t_{k-1}}(\omega)| \\ &= \sup \sum_{k=1}^n |X_{t_k\wedge T(\omega)}(\omega) - X_{t_{k-1}\wedge T(\omega)}(\omega)| \le (V_X)_{t\wedge T(\omega)}(\omega).\end{aligned}$$

Combining our conclusions, the result follows. □

Lemma 1.2.11. *Assume that $M \in \mathcal{M}$ is almost surely continuous and has paths of finite variation. Then M is evanescent.*

Proof. Let $M \in \mathcal{M}$ be almost surely continuous and have paths of finite variation. Let F be the null set where M is not continuous and put $N = 1_{F^c} M$. As the usual conditions hold,

$F \in \mathcal{F}_0 \subseteq \mathcal{F}$. Therefore, $N \in \mathcal{M}$ as well, N has paths of finite variation and N is continuous. And if N is evanescent, M is evanescent as well. We conclude that it suffices to prove the theorem in the case where $M \in \mathbf{c}\mathcal{M}$ and M has paths of finite variation.

We first consider the case where $M \in \mathbf{c}\mathcal{M}^b$ and the variation process V_M is bounded, $(V_M)_t$ being the variation of M over $[0,t]$. Fix $t \geq 0$ and let $t_k^n = kt2^{-n}$. Now note that by the martingale property, $EM_{t_{k-1}^n}(M_{t_k^n} - M_{t_{k-1}^n}) = EM_{t_{k-1}^n}E(M_{t_k^n} - M_{t_{k-1}^n}|\mathcal{F}_{t_{k-1}^n}) = 0$, and by rearrangement, $M_{t_k^n}^2 - M_{t_{k-1}^n}^2 = 2M_{t_{k-1}^n}(M_{t_k^n} - M_{t_{k-1}^n}) + (M_{t_k^n} - M_{t_{k-1}^n})^2$. Therefore, we obtain

$$\begin{aligned} EM_t^2 &= E\sum_{k=1}^{2^n}(M_{t_k^n}^2 - M_{t_{k-1}^n}^2) = 2E\sum_{k=1}^{2^n} M_{t_{k-1}^n}(M_{t_k^n} - M_{t_{k-1}^n}) + E\sum_{k=1}^{2^n}(M_{t_k^n} - M_{t_{k-1}^n})^2 \\ &= E\sum_{k=1}^{2^n}(M_{t_k^n} - M_{t_{k-1}^n})^2 \leq E(V_M)_t \max_{k\leq 2^n}|M_{t_k^n} - M_{t_{k-1}^n}|. \end{aligned}$$

Now, as M is continuous, $(V_M)_t \max_{k\leq n}|M_{t_k^n} - M_{t_{k-1}^n}|$ tends pointwisely to zero as n tends to infinity. The boundedness of M and V_M then allows us to apply the dominated convergence theorem and obtain

$$EM_t^2 \leq \lim_{n\to\infty} E(V_M)_t \max_{k\leq 2^n}|M_{t_k} - M_{t_{k-1}}| \leq E \lim_{n\to\infty}(V_M)_t \max_{k\leq 2^n}|M_{t_k} - M_{t_{k-1}}| = 0,$$

so that M_t is almost surely zero by Lemma A.1.20, and so by Lemma 1.1.7, M is evanescent. In the case of a general $M \in \mathbf{c}\mathcal{M}$, define $T_n = \inf\{t \geq 0 \mid |(V_M)_t| > n\}$. By Lemma 1.1.17, (T_n) is a sequence of stopping times increasing almost surely to infinity, and $(V_M)^{T_n}$ is bounded by n. By Lemma A.2.10, $|M_t^{T_n}| \leq |(V_M)_t^{T_n}| \leq n$ for all $t \geq 0$. As $(V_M)^{T_n} = V_{M^{T_n}}$ by Lemma 1.2.10, M^{T_n} is a bounded martingale with bounded variation, so our previous results show that M^{T_n} is evanescent. Letting n tend to infinity, T_n tends to infinity, and so we almost surely obtain $M_t = \lim_n M_t^{T_n} = 0$, allowing us to conclude by Lemma 1.1.7 that M is evanescent. □

We end the section by introducing two types of processes which will serve as instructive examples for most of the theory to follow. Often, properties of these two types of processes may be elegantly derived by the application of martingale methods. First, we introduce the concept of an $\mathcal{F}_t$ Brownian motion.

Definition 1.2.12. *A p-dimensional $\mathcal{F}_t$ Brownian motion is a continuous process W adapted to $\mathcal{F}_t$ such that for any t, the distribution of $s \mapsto W_{t+s} - W_t$ is a p-dimensional Brownian motion independent of $\mathcal{F}_t$.*

Essentially, the difference between a plain p-dimensional Brownian motion and a p-dimensional

$\mathcal{F}_t$ Brownian motion is that the p-dimensional $\mathcal{F}_t$ Brownian motion possesses a certain regular relationship with the filtration. The independence in Definition 1.2.12 means the following. Fix $t \geq 0$ and let X be the process defined by $X_s = W_{t+s} - W_t$. The process X is then a random variable with values in $C(\mathbb{R}_+, \mathbb{R}^p)$, the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^p$, endowed with the σ-algebra $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^p)$ induced by the coordinate mappings. The independence of X and $\mathcal{F}_t$ means that for any $A \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^p)$ and any $B \in \mathcal{F}_t$, $P((X \in A) \cap B) = P(X \in A)P(B)$.

The following basic result shows that the martingales associated with ordinary Brownian motions reoccur when considering $\mathcal{F}_t$ Brownian motions.

Theorem 1.2.13. *Let W be a p-dimensional $\mathcal{F}_t$ Brownian motion. For $i \leq p$, W^i and $(W_t^i)^2 - t$ are martingales, where W^i denotes the i'th coordinate of W. For $i, j \leq p$ with $i \neq j$, $W_t^i W_t^j$ is a martingale.*

Proof. Let $i \leq p$ and let $0 \leq s \leq t$. W^i is then an $\mathcal{F}_t$ Brownian motion, so $W_t^i - W_s^i$ is normally distributed with mean zero and variance $t - s$ and independent of $\mathcal{F}_t$. Therefore, we obtain $E(W_t^i|\mathcal{F}_s) = E(W_t^i - W_s^i|\mathcal{F}_s) + W_s^i = E(W_t^i - W_s^i) + W_s^i = W_s^i$, proving that W^i is a martingale. Furthermore, we find

$$\begin{aligned} E((W_t^i)^2 - t|\mathcal{F}_s) &= E((W_t^i - W_s^i)^2 - (W_s^i)^2 + 2W_s^i W_t^i|\mathcal{F}_s) - t \\ &= E((W_t^i - W_s^i)^2|\mathcal{F}_s) - (W_s^i)^2 + 2W_s^i E(W_t^i|\mathcal{F}_s) - t = (W_s^i)^2 - s, \end{aligned}$$

so $(W_t^i)^2 - t$ is a martingale. Next, let $i, j \leq p$ with $i \neq j$. We then obtain that for $0 \leq s \leq t$, using independence and the martingale property,

$$\begin{aligned} E(W_t^i W_t^j|\mathcal{F}_s) &= E(W_t^i W_t^j - W_s^i W_s^j|\mathcal{F}_s) + W_s^i W_s^j \\ &= E(W_t^i W_t^j - W_t^i W_s^j + W_t^i W_s^j - W_s^i W_s^j|\mathcal{F}_s) + W_s^i W_s^j \\ &= E(W_t^i(W_t^j - W_s^j) - W_s^j(W_t^i - W_s^i)|\mathcal{F}_s) + W_s^i W_s^j \\ &= E(W_t^i(W_t^j - W_s^j)|\mathcal{F}_s) + W_s^i W_s^j \\ &= E((W_t^i - W_s^i)(W_t^j - W_s^j)|\mathcal{F}_s) + E(W_s^i(W_t^j - W_s^j)|\mathcal{F}_s) + W_s^i W_s^j \\ &= W_s^i W_s^j, \end{aligned}$$

where we have used that the variables $E(W_s^j(W_t^i - W_s^i)|\mathcal{F}_s)$, $E((W_t^i - W_s^i)(W_t^j - W_s^j)|\mathcal{F}_s)$ and $E(W_s^i(W_t^j - W_s^j)|\mathcal{F}_s)$ are all equal to zero, because $s \mapsto W_{t+s} - W_t$ is independent of $\mathcal{F}_s$ and has the distribution of a p-dimensional Brownian motion. Thus, $W^i W^j$ is a martingale. □

Furthermore, when W is a p-dimensional $\mathcal{F}_t$ Brownian motion, W^i has the distribution of a Brownian motion, so all ordinary distributional results for Brownian motion transfer verbatim to $(\mathcal{F}_t)$ Brownian motion, for example that the following results hold almost surely:

$$\limsup_{t\to\infty} \frac{W_t^i}{\sqrt{2t\log\log t}} = 1, \liminf_{t\to\infty} \frac{W_t^i}{\sqrt{2t\log\log t}} = -1, \lim_{t\to\infty} \frac{W_t^i}{t} = 0.$$

Next, we introduce $(\mathcal{F}_t)$ Poisson processes in a manner similar to that of $(\mathcal{F}_t)$ Brownian motions.

Definition 1.2.14. *An $\mathcal{F}_t$ Poisson process is an increasing càdlàg process N adapted to $\mathcal{F}_t$ such that $N_t = \sum_{0<s\leq t} \Delta N_s$, where ΔN only takes the values zero and one, and such that for any t, the distribution of $s \mapsto N_{t+s} - N_t$ is a Poisson process independent of $\mathcal{F}_t$.*

Similarly to Definition 1.2.12, the difference between an $\mathcal{F}_t$ Poisson process and a Poisson process is that an an $\mathcal{F}_t$ Poisson process has regularity propeties ensuring that the process interacts properly with the filtration of the probability space.

Theorem 1.2.15. *Let N be an $\mathcal{F}_t$ Poisson process. Then $N_t - t$ is a martingale.*

Proof. Let $0 \leq s \leq t$. Using independence, we obtain

$$E(N_t - t|\mathcal{F}_s) = E(N_t - N_s|\mathcal{F}_s) + N_s - t = E(N_t - N_s) + N_s - t = N_s - s,$$

which proves the result. $\square$

Also, note the following. Letting N be an $\mathcal{F}_t$ Poisson process, we may define $T_0 = 0$ and for $n \geq 1$, $T_n = \inf\{t \geq 0 \mid N_t = n\}$. As N is càdlàg, is the sum of its jumps and only has jumps of size one, it holds that the sequence (T_n) is strictly increasing and covers the jumps of N. As we also have $T_n = \inf\{t \geq 0 \mid N_t > n - 1/2\}$, Lemma 1.1.13 shows that each T_n is a stopping time. And as N has the distribution of a Poisson process, the sequence $(T_n - T_{n-1})_{n\geq 1}$ is is and independent and identically distributed sequence of standard exponentially distributed variables, and we have $N_t = \sum_{n=1}^{\infty} 1_{(t\leq T_n)}$.

1.3 Square-integrable martingales

In this section, we consider the properties of square-integrable martingales, and we apply these properties to prove the existence of the quadratic variation process for bounded martingales. We say that a martingale M is square-integrable if $\sup_{t\geq 0} EM_t^2$ is finite. The space of càdlàg square-integrable martingales with initial value zero is denoted by $\mathcal{M}^2$. It holds that $\mathcal{M}^2$ is a vector space. By $\mathbf{c}\mathcal{M}^2$, we denote the space of continuous elements of $\mathcal{M}^2$.

For the following theorem, we introduce some further notation. For any process X, we put $X_t^* = \sup_{s\le t}|X_s|$ and $X_\infty^* = \sup_{t\ge 0}|X_t|$. We write $X_t^{*2} = (X_t^*)^2$, and likewise $X_\infty^{*2} = (X_\infty^*)^2$.

Theorem 1.3.1. *Let $M \in \mathcal{M}^2$. Then, there exists a square-integrable variable M_∞ such that $M_t = E(M_\infty|\mathcal{F}_t)$ for all $t \ge 0$. Furthermore, M_t converges to M_∞ almost surely and in $\mathcal{L}^2$, and $EM_\infty^{*2} \le 4EM_\infty^2$.*

Proof. As M is bounded in $\mathcal{L}^2$, M is in particular uniformly integrable by Lemma A.3.4, so by Theorem 1.2.4, M_t converges almost surely and in $\mathcal{L}^1$ to some variable M_∞, which is integrable and satisfies that $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely for $t \ge 0$. It remains to prove that M_∞ is square-integrable, that we have convergence in $\mathcal{L}^2$ and that $EM_\infty^{*2} \le 4EM_\infty^2$ holds.

Put $t_k^n = k2^{-n}$ for $n, k \ge 0$. Then $(M_{t_k^n})_{k\ge 0}$ is a discrete-time martingale for $n \ge 0$ with $\sup_{k\ge 0} EM_{t_k^n}^2$ finite. By Lemma A.4.4, $M_{t_k^n}$ converges almost surely and in $\mathcal{L}^2$ to some square-integrable limit as k tends to infinity. By uniqueness of limits, the limit is M_∞, so we conclude that M_∞ is square-integrable. Lemma A.4.4 also yields $E\sup_{k\ge 0} M_{t_k^n}^2 \le 4EM_\infty^2$. We then obtain by the monotone convergence theorem and the right-continuity of M that

$$EM_\infty^{*2} = E\lim_{n\to\infty}\sup_{k\ge 0} M_{t_k^n}^2 = \lim_{n\to\infty} E\sup_{k\ge 0} M_{t_k^n}^2 \le 4EM_\infty^2.$$

This proves the inequality $EM_\infty^{*2} \le 4EM_\infty^2$. It remains to show that M_t converges to M_∞ in $\mathcal{L}^2$. To this end, note that as we have $(M_t - M_\infty)^2 \le (2M_\infty^*)^2 = 4M_\infty^{*2}$, which is integrable, the dominated convergence theorem yields $\lim_t E(M_t - M_\infty)^2 = E\lim_t (M_t - M_\infty)^2 = 0$, so M_t also converges in $\mathcal{L}^2$ to M_∞, as desired. □

Lemma 1.3.2. *Assume that $M \in \mathcal{M}^2$. Then $M^T \in \mathcal{M}^2$ as well.*

Proof. By Lemma 1.2.7, M^T is a martingale. Furthermore, we have

$$\sup_{t\ge 0} E(M^T)_t^2 \le E\sup_{t\ge 0}(M_t^T)^2 \le E\sup_{t\ge 0} M_t^2 = EM_\infty^{*2},$$

and this is finite by Theorem 1.3.1, proving that $M^T \in \mathcal{M}^2$. □

Theorem 1.3.3. *Assume that (M^n) is a sequence in $\mathcal{M}^2$ such that (M_∞^n) is convergent in $\mathcal{L}^2$ to a limit M_∞. Then there is some $M \in \mathcal{M}^2$ such that for all $t \ge 0$, $M_t = E(M_\infty|\mathcal{F}_t)$. Furthermore, $E\sup_{t\ge 0}(M_t^n - M_t)^2$ tends to zero.*

Proof. The difficulty in the proof lies in demonstrating that the martingale M obtained by putting $M_t = E(M_\infty|\mathcal{F}_t)$ has a càdlàg version. First note that $M^n - M^m \in \mathcal{M}^2$ for all n and m, so for any $\delta > 0$ we may apply Chebychev's inequality and Theorem 1.3.1 to obtain, using $(x+y)^2 \leq 4x^2 + 4y^2$,

$$\begin{aligned} P((M^n - M^m)^*_\infty \geq \delta) &\leq \delta^{-2} E(M^n - M^m)^{*2}_\infty \\ &\leq 4\delta^{-2} E(M^n_\infty - M^m_\infty)^2 \\ &\leq 16\delta^{-2}(E(M^n_\infty - M_\infty)^2 + E(M_\infty - M^m_\infty)^2) \\ &\leq 32\delta^{-2} \sup_{k \geq m \wedge n} E(M^k_\infty - M_\infty)^2. \end{aligned}$$

Now let (n_i) be a strictly increasing sequence of naturals such that for each i, it holds that

$$32(2^{-i})^{-2} \sup_{k \geq n_i} E(M^k_\infty - M_\infty)^2 \leq 2^{-i},$$

this is possible as $\sup_{k \geq n} E(M^k_\infty - M_\infty)^2$ tends to zero as n tends to infinity. In particular, $P((M^{n_{i+1}} - M^{n_i})^*_\infty > 2^{-i}) \leq 2^{-i}$ for all $i \geq 1$. Then $\sum_{i=1}^\infty P((M^{n_{i+1}} - M^{n_i})^*_\infty > 2^{-i})$ is finite, so therefore, by the Borel-Cantelli Lemma, the event that $(M^{n_{i+1}} - M^{n_i})^*_\infty > 2^{-i}$ infinitely often has probability zero. Therefore, $(M^{n_{i+1}} - M^{n_i})^*_\infty \leq 2^{-i}$ from a point onwards almost surely. In particular, it almost surely holds that for any two numbers $k \leq m$ large enough, depending on ω,

$$(M^{n_m} - M^{n_k})^*_\infty \leq \sum_{i=k+1}^{m} (M^{n_i} - M^{n_{i-1}})^*_\infty \leq \sum_{i=k+1}^{\infty} 2^{-i} = 2^{-k}.$$

Thus, it holds almost surely that M^{n_i} is Cauchy in the uniform norm on $\mathbb{R}_+$, and therefore by Lemma A.2.6 almost surely uniformly convergent to some càdlàg limit. Define M to be the uniform limit when it exists and zero otherwise, M is then a process with càdlàg paths. With F being the null set where we have uniform convergence, our assumption that the usual conditions hold allows us to conclude that $F \in \mathcal{F}_t$ for all $t \geq 0$. As uniform convergence implies pointwise convergence, we have $M_t = 1_F \lim_{i\to\infty} M^{n_i}_t$, so M is also adapted. We now claim that $M \in \mathcal{M}^2$. To see this, note that by Jensen's inequality, we have

$$\begin{aligned} E(M^n_t - E(M_\infty|\mathcal{F}_t))^2 &= E(E(M^n_\infty|\mathcal{F}_t) - E(M_\infty|\mathcal{F}_t))^2 = EE(M^n_\infty - M_\infty|\mathcal{F}_t)^2 \\ &\leq EE((M^n_\infty - M_\infty)^2|\mathcal{F}_t) = E(M^n_\infty - M_\infty)^2, \end{aligned}$$

which tends to zero, so for any $t \geq 0$, M^n_t tends to $E(M_\infty|\mathcal{F}_t)$ in $\mathcal{L}^2$. As $M^{n_k}_t$ tends to M_t almost surely, we conclude that $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely by uniqueness of limits. This shows that M is a martingale, and as $EM_t^2 \leq EE(M^2_\infty|\mathcal{F}_t) = EM^2_\infty$, which is finite, we conclude that M is bounded in $\mathcal{L}^2$. As M clearly has initial value zero, we then obtain $M \in \mathcal{M}^2$. Finally, $\limsup_n E\sup_{t\geq 0}(M^n_t - M_t)^2 \leq 4\lim_n E(M^n_\infty - M_\infty)^2 = 0$ by Theorem 1.3.1, yielding the desired convergence of M^n to M. □

We now introduce a seminorm $\|\cdot\|_2$ on the space $\mathcal{M}^2$ by putting $\|M\|_2 = (EM_\infty^2)^{\frac{1}{2}}$, this is possible as we have ensured in Theorem 1.3.1 that for any $M \in \mathcal{M}^2$, $M_t = E(M_\infty|\mathcal{F}_t)$ for some almost surely unique square-integrable M_∞, so that the limit determines the entire martingale. Note that $\|\cdot\|_2$ is generally not a norm, only a seminorm, in the sense that $\|M\|_2 = 0$ does not imply that M is zero, only that M is evanescent.

Theorem 1.3.4. *The space $\mathcal{M}^2$ is complete under the seminorm $\|\cdot\|$, in the sense that any Cauchy sequence in $\mathcal{M}^2$ has a limit.*

Proof. Assume that (M^n) is a Cauchy sequence in $\mathcal{M}^2$. By our definition of the seminorm on $\mathcal{M}^2$, we have $(E(M_\infty^n - M_\infty^m)^2)^{\frac{1}{2}} = \|M^n - M^m\|_2$, and so (M_∞^n) is a Cauchy sequence in $\mathcal{L}^2$. As $\mathcal{L}^2$ is complete, there exists M_∞ such that M_∞^n converges in $\mathcal{L}^2$ to M_∞. By Theorem 1.3.3, there exists $M \in \mathcal{M}^2$ such that for any $t \geq 0$, $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely. Therefore, M^n tends to M in $\mathcal{M}^2$. □

Theorem 1.3.5 (Riesz' representation theorem for $\mathcal{M}^2$)**.** *Let $M \in \mathcal{M}^2$. Then, the mapping $\varphi : \mathcal{M}^2 \to \mathbb{R}$ defined by $\varphi(N) = EM_\infty N_\infty$ is linear and continuous. Conversely, if it holds that $\varphi : \mathcal{M}^2 \to \mathbb{R}$ is linear and continuous, there exists $M \in \mathcal{M}^2$, unique up to indistinguishability, such that $\varphi(N) = EM_\infty N_\infty$ for all $N \in \mathcal{M}^2$.*

Proof. First consider $M \in \mathcal{M}^2$ and define $\varphi : \mathcal{M}^2 \to \mathbb{R}$ by putting $\varphi(N) = EM_\infty N_\infty$. The mapping φ is then clearly linear, and $|\varphi(N - N')| = |EM_\infty(N_\infty - N'_\infty)| \leq \|M\|_2\|N - N'\|_2$ for all $N, N' \in \mathcal{M}^2$ by the Cauchy-Schwartz inequality, showing that φ is Lipschitz with Lipschitz constant $\|M\|_2$, therefore continuous.

Conversely, assume given any $\varphi : \mathcal{M}^2 \to \mathbb{R}$ which is linear and continuous, we need to find $M \in \mathcal{M}^2$ such that $\varphi(N) = EM_\infty N_\infty$ for all $N \in \mathcal{M}^2$. If φ is identically zero, this is trivially satisfied with M being the zero martingale. Therefore, assume that φ is not identically zero. In this case, there is $M' \in \mathcal{M}^2$ such that $\varphi(M') \neq 0$. Define the set $C \subseteq \mathcal{M}^2$ by $C = \{L \in \mathcal{M}^2 \mid \varphi(L) = \|M'\|_2\}$. As φ is continuous, C is closed. And as φ is linear, C is convex.

We claim that there is $M'' \in C$ such that such that $\|M''\|_2 = \inf_{L\in C}\|L\|_2$. To prove this, it suffices to put $\alpha = \inf_{L\in C}\|L\|_2^2$ and identify $M'' \in C$ such that $\|M''\|_2^2 = \alpha$. Take a sequence (L^n) in C such that $\|L^n\|^2$ converges to α. Since $\frac{1}{2}(L^m + L^n) \in C$ by convexity, we have

$$\begin{aligned} \|L^m - L^n\|_2^2 &= 2\|L^m\|_2^2 + 2\|L^n\|_2^2 - \|L^m + L^n\|_2^2 \\ &= 2\|L^m\|_2^2 + 2\|L^n\|_2^2 - 4\|\tfrac{1}{2}(L^m + L^n)\|_2^2 \leq 2\|L^m\|_2^2 + 2\|L^n\|_2^2 - 4\alpha. \end{aligned}$$

As m and n tend to infinity, $\|L^m\|_2^2$ and $\|L^n\|_2^2$ tend to α, so $\|L^m - L^n\|_2^2$ tends to zero. Therefore, (L^n) is Cauchy. By Theorem 1.3.4, (L^n) is convergent towards some M''. As C is closed, $M'' \in C$, and we furthermore find $\|M''\|_2^2 = \lim_n \|L^n\|_2^2 = \alpha$, as desired.

We next claim that for any $N \in \mathcal{M}^2$ with $\varphi(N) = 0$, $EM''_\infty N_\infty = 0$. This is clearly true if N is evanescent, assume therefore that N is not evanescent, so that $\|N\|_2 \neq 0$. By linearity, $\varphi(M'' - tN) = \varphi(M'')$ for any $t \in \mathbb{R}$, so that $M'' - tN \in C$. We then find $\|M''\|_2^2 = \inf_{L\in C} \|L\|_2^2 \leq \inf_{t\in\mathbb{R}} \|M'' - tN\|_2^2 \leq \|M''\|_2^2$, so that $\|M''\|_2^2$ is the minimum of the mapping $t \mapsto \|M'' - tN\|_2^2$, attained at zero. However, we also have the relation $\|M''-tN\|_2^2 = t^2\|N\|_2^2 - 2tEM''_\infty N_\infty + \|M\|_2^2$, so $t \mapsto \|M''_\infty - tN_\infty\|_2^2$ is a quadratic polynomial, and as $\|N\|_2 \neq 0$, it attains its unique minimum at $\|N\|_2^{-2}EM''_\infty N_\infty$. As we also know that the minimum is attained at zero, we conclude $EM''_\infty N_\infty = 0$.

We have now proven the existence of a process M'' in $\mathcal{M}^2$ which is nonzero and satisfies $EM''_\infty N_\infty = 0$ whenever $\varphi(N) = 0$. We then note for any $N \in \mathcal{M}^2$ that, using the linearity of φ, $\varphi(\varphi(M'')N - \varphi(N)M'') = \varphi(M'')\varphi(N) - \varphi(N)\varphi(M'') = 0$, yielding the relationship $0 = EM''_\infty(\varphi(M'')N_\infty - \varphi(N)M''_\infty) = \varphi(M'')EM''_\infty N_\infty - \varphi(N)\|M''\|_2^2$, so that we finally obtain the relation

$$\varphi(N) = \|M''\|_2^{-2}\varphi(M'')EM''_\infty N_\infty = E\left(\frac{\varphi(M'')M''_\infty}{\|M''\|_2^2}N_\infty\right),$$

which proves the desired result using the element $(\varphi(M'')M'')\|M''\|_2^{-2}$ of $\mathcal{M}^2$. It remains to prove uniqueness. Assume therefore that $M, M' \in \mathcal{M}^2$ such that $EM_\infty N_\infty = EM'_\infty N_\infty$ for all $N \in \mathcal{M}^2$. Then $E(M_\infty - M'_\infty)N_\infty = 0$ for all $N \in \mathcal{M}^2$, in particular we have $E(M_\infty - M'_\infty)^2 = 0$ so that $M_\infty = M'_\infty$ almost surely and so M and M' are indistinguishable. This completes the proof. $\square$

Finally, we apply our results on $\mathcal{M}^2$ to prove the existence of the quadratic variation process for bounded martingales. We say that a process X is increasing if its sample paths are increasing. In this case, the limit of X_t exists almost surely as a variable with values in $[0, \infty]$ and is denoted by X_∞. We say that an increasing process is integrable if its limit X_∞ is integrable. In this case, X_∞ is in particular almost surely finite. We denote by $\mathcal{A}^i$ the set of stochastic processes with initial value zero which are càdlàg, adapted, increasing, and integrable.

Theorem 1.3.6. *Let $M \in \mathcal{M}^b$. There exists a process $[M]$ in $\mathcal{A}^i$, unique up to indistinguishability, such that $M^2 - [M] \in \mathcal{M}^2$ and such that $\Delta[M] = (\Delta M)^2$ almost surely. We call $[M]$ the quadratic variation process of M.*

Proof. We first consider uniqueness. Assume that A and B are two processes in $\mathcal{A}^i$ such that $M^2 - A$ and $M^2 - B$ are in $\mathcal{M}^2$ and $\Delta A = \Delta B = (\Delta M)^2$ almost surely. In particular, $A - B$ is in $\mathcal{M}^2$, is almost surely continuous and has paths of finite variation, so Lemma 1.2.11 shows that $A - B$ is evanescent, such that A and B are indistinguishable. This proves uniqueness.

Next, we consider the existence of the process. Let $t_k^n = k2^{-n}$ for $n, k \geq 0$, we then find

$$M_t^2 = \sum_{k=1}^{\infty} M_{t\wedge t_k^n}^2 - M_{t\wedge t_{k-1}^n}^2 = 2\sum_{k=1}^{\infty} M_{t\wedge t_{k-1}^n}(M_{t\wedge t_k^n} - M_{t\wedge t_{k-1}^n}) + \sum_{k=1}^{\infty}(M_{t\wedge t_k^n} - M_{t\wedge t_{k-1}^n})^2,$$

where the terms in the sum are zero from a point onwards, namely for such k that $t_{k-1}^n \geq t$. Define $N_t^n = 2\sum_{k=1}^{\infty} M_{t\wedge t_{k-1}^n}(M_{t\wedge t_k^n} - M_{t\wedge t_{k-1}^n})$. Our plan for the proof is to show that N^n is in $\mathcal{M}^2$ and that $(N_\infty^n)_{n\geq 1}$ is bounded in $\mathcal{L}^2$. This will allow us to apply Lemma A.3.7 in order to obtain some $N \in \mathcal{M}^2$ which is the limit of appropriate convex combinations of the (N^n). We then show that by putting $[M] = M^2 - N$, we obtain, up to indistinguishability, a process with the desired qualities.

We first show that $N^n \in \mathcal{M}$ by applying Lemma 1.2.8. Clearly, N^n is càdlàg and adapted with initial value zero, and so it suffices to prove that N_T^n is integrable and that $EN_T^n = 0$ for all bounded stopping times T. To this end, note that as M is bounded, there is $c > 0$ such that $|M_t| \leq c$ for all $t \geq 0$. Therefore, $|2M_{t\wedge t_{k-1}^n}(M_{t\wedge t_k^n} - M_{t\wedge t_{k-1}^n})| \leq 4c^2$ for any k. As T is also bounded, N_T^n is integrable, as it is the sum of finitely many terms bounded by $4c^2$, and the martingale property of M^T yields

$$\begin{aligned} EN_T^n &= E\sum_{k=1}^{\infty} M_{T\wedge t_{k-1}^n}(M_{T\wedge t_k^n} - M_{T\wedge t_{k-1}^n}) \\ &= \sum_{k=1}^{\infty} EM_{t_{k-1}^n}^T(M_{t_k^n}^T - M_{t_{k-1}^n}^T) = \sum_{k=1}^{\infty} EM_{t_{k-1}^n}^T E(M_{t_k^n}^T - M_{t_{k-1}^n}^T|\mathcal{F}_{t_{k-1}^n}) = 0, \end{aligned}$$

where the interchange of summation and expectation is allowed, as the only nonzero terms in the sum are for those k such that $t_{k-1}^n \leq T$, and there are only finitely many such terms. Thus, by Lemma 1.2.8, $N^n \in \mathcal{M}$.

Next, we show that $(N_\infty^n)_{n\geq 1}$ is bounded in $\mathcal{L}^2$. Fix $k \geq 1$, we first consider a bound for the second moment of $N_{t_k^n}^n$. To obtain this, note that for $i < j$,

$$\begin{aligned} & EM_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n})M_{t_{j-1}^n}(M_{t_j^n} - M_{t_{j-1}^n}) \\ = \; & E(M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n})E(M_{t_{j-1}^n}(M_{t_j^n} - M_{t_{j-1}^n})|\mathcal{F}_{t_i^n})) \\ = \; & E(M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n})M_{t_{j-1}^n}E(M_{t_j^n} - M_{t_{j-1}^n}|\mathcal{F}_{t_i^n})), \end{aligned}$$

which is zero, as $E((M_{t_j^n} - M_{t_{j-1}^n})|\mathcal{F}_{t_i^n}) = 0$, and by the same type of argument, we obtain $E(M_{t_i^n} - M_{t_{i-1}^n})(M_{t_j^n} - M_{t_{j-1}^n}) = 0$. Therefore, we obtain

$$\begin{aligned} E(N^n_{t_k^n})^2 &= E\left(\sum_{i=1}^k M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n})\right)^2 = \sum_{i=1}^k E\left(M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n})\right)^2 \\ &\le c^2 \sum_{i=1}^k E(M_{t_i^n} - M_{t_{i-1}^n})^2 = c^2 E\left(\sum_{i=1}^k M_{t_i^n} - M_{t_{i-1}^n}\right)^2 = c^2 E M^2_{t_k^n}. \end{aligned}$$

Now, finally note that for any $0 \le s \le t$, $E(N_s^n)^2 = E(E(N_t^n|\mathcal{F}_s)^2) \le E(N_t^n)^2$ by Jensen's inequality, so $t \mapsto E(N_t^n)^2$ is increasing, and as we also have $EM^2_{t_k} \le EM^2_\infty$ for all $k \ge 1$, we get $\sup_{t\ge 0} E(N_t^n)^2 = \sup_{k\ge 1} E(N^n_{t_k^n})^2 \le \sup_{k\ge 1} c^2 EM^2_{t_k^n} \le 4c^2 EM^2_\infty$, and the latter is finite. Thus, $N^n \in \mathcal{M}^2$, and in particular, $E(N_\infty^n)^2 = \lim_t E(N_t^n)^2 \le 4c^2 EM^2_\infty$, so $(N^n_\infty)_{n\ge 1}$ is bounded in $\mathcal{L}^2$.

Now, by Lemma A.3.7, there exists a sequence of naturals (K_n) with $K_n \ge n$ and for each n a finite sequence of reals $\lambda_n^n, \ldots, \lambda_{K_n}^n$ in the unit interval summing to one, such that $\sum_{i=n}^{K_n} \lambda_i^n N_\infty^i$ is convergent in $\mathcal{L}^2$ to some variable N_∞. By Theorem 1.3.3, it then holds that there is $N \in \mathcal{M}^2$ such that $E\sup_{t\ge 0}(N_t - \sum_{i=n}^{K_n} \lambda_i^n N_t^i)^2$ tends to zero. Picking a subsequence and relabeling, we may in addition to the properties already noted assume that $\sup_{t\ge 0}|N_t - \sum_{i=n}^{K_n} \lambda_i^n N_t^i|$ also converges almost surely to zero. Define $A = M^2 - N$, we claim that there is a modification of A satisfying the criteria of the theorem.

To prove this, first note that as M^2 and N are càdlàg and adapted, so is A. We want to show that A is almost surely increasing, that the almost sure limit A_∞ is integrable and that $\Delta A = (\Delta M)^2$ almost surely. We first consider the jumps of A. To prove that $\Delta A = (\Delta M)^2$ almost surely, it suffices by Lemma 1.1.16 to show that $\Delta A_T = (\Delta M_T)^2$ almost surely for any bounded stopping time T. Let T be any bounded stopping time. Since $\sup_{t\ge 0}|N_t - \sum_{i=n}^{K_n} \lambda_i^n N_t^i|$ converges almost surely to zero, we find

$$A_t = M_t^2 - N_t = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n (M_t^2 - N_t^i) = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^\infty (M_t^{t_k^i} - M_t^{t_{k-1}^i})^2,$$

with the limits being almost sure, uniformly over $t \ge 0$. In particular, we obtain

$$\Delta A_T = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^\infty (M_T^{t_k^i} - M_T^{t_{k-1}^i})^2 - (M_{T-}^{t_k^i} - M_{T-}^{t_{k-1}^i})^2,$$

again, the limit being almost sure. Fix $i \geq 1$ and $k \geq 0$. Note that

$$\begin{aligned}
(M_t^{t_k^i} - M_t^{t_{k-1}^i})^2 - (M_{t-}^{t_k^i} - M_{t-}^{t_{k-1}^i})^2 = 0 & \qquad \text{when } t \leq t_{k-1}^i \text{ or } t > t_k^i \\
(M_t^{t_k^i} - M_t^{t_{k-1}^i})^2 = (M_t - M_{t_{k-1}^i})^2 & \qquad \text{when } t_{k-1}^i < t \leq t_k^i \\
(M_{t-}^{t_k^i} - M_{t-}^{t_{k-1}^i})^2 = (M_{t-} - M_{t_{k-1}^i})^2 & \qquad \text{when } t_{k-1}^i < t \leq t_k^i.
\end{aligned}$$

From these observations, we conclude that with $s(t,i)$ denoting the unique t_{k-1}^i such that $t_{k-1}^i < t \leq t_k^i$, we have $\Delta A_T = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n ((M_T - M_{s(T,i)})^2 - (M_{T-} - M_{s(T,i)})^2)$. Here, it holds that

$$\begin{aligned}
& (M_T - M_{s(T,i)})^2 - (M_{T-} - M_{s(T,i)})^2 \\
= \;& M_T^2 - 2M_T M_{s(T,i)} + M_{s(T,i)}^2 - (M_{T-}^2 - 2M_{T-}M_{s(T,i)} + M_{s(T,i)}^2) \\
= \;& M_T^2 - M_{T-}^2 - 2\Delta M_T M_{s(T,i)} = (M_T - M_{T-})(M_T + M_{T-}) - 2\Delta M_T M_{s(T,i)} \\
= \;& (\Delta M_T)^2 + 2\Delta M_T (M_{T-} - M_{s(T,i)}),
\end{aligned}$$

yielding $\Delta A_T = (\Delta M_T)^2 + 2\Delta M_T \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n (M_{T-} - M_{s(T,i)})$. Now, we always have $s(T,i) < T$ and $|s(T,i) - T| \leq 2^{-i}$. Therefore, given $\varepsilon > 0$, there is $n \geq 1$ such that for all $i \geq n$, $|M_{T-} - M_{s(T,i)}| \leq \varepsilon$. As the $(\lambda_i^n)_{n \leq i \leq K_n}$ are convex weights, we obtain for n this large that $|\sum_{i=n}^{K_n} \lambda_i^n (M_{T-} - M_{s(T,i)})| \leq \varepsilon$. This allows us to conclude that $\sum_{i=n}^{K_n} \lambda_i^n (M_{T-} - M_{s(T,i)})$ converges almost surely to zero. Combining this with our previous conclusions, we obtain $\Delta A_T = (\Delta M_T)^2$ almost surely. Since this holds for any arbitrary stopping time, we now obtain $\Delta A = (\Delta M)^2$ up to indistinguishability.

Next, we show that A is almost surely increasing. Put $\mathbb{D}_+ = \{k2^{-n} | k \geq 0, n \geq 1\}$, $\mathbb{D}_+$ is dense in $\mathbb{R}_+$. Let $p, q \in \mathbb{D}_+$ with $p \leq q$, we will show that $A_p \leq A_q$ almost surely. There exists $j \geq 1$ and naturals $n_p \leq n_q$ such that $p = n_p 2^{-j}$ and $q = n_q 2^{-j}$. By what we already have shown, we then find, with the limit being almost sure, that

$$A_p = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{\infty} (M_p^{t_k^i} - M_p^{t_{k-1}^i})^2.$$

Now note that for $i \geq j$, we have $p \wedge t_k^i = n_p 2^{-j} \wedge k2^{-i} = n_p 2^{i-j} 2^{-i} \wedge k 2^{-i} = (n_p 2^{i-j} \wedge k) 2^{-i}$ and analogously for $q \wedge t_k^i$, so we obtain that almost surely,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{\infty} (M_p^{t_k^i} - M_p^{t_{k-1}^i})^2 &= \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{n_p 2^{i-j}} (M_{t_k^i} - M_{t_{k-1}^i})^2 \\
&\leq \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{n_q 2^{i-j}} (M_{t_k^i} - M_{t_{k-1}^i})^2 \\
&= \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^m \sum_{k=1}^{\infty} (M_q^{t_k^i} - M_q^{t_{k-1}^i})^2,
\end{aligned}$$

allowing us to make the same calculations in reverse and conclude that $A_p \leq A_q$ almost surely. As $\mathbb{D}_+$ is countable, we conclude that A is inceasing on $\mathbb{D}_+$ almost surely, and by continuity, we conclude that A is increasing almost surely. Furthermore, as $A_\infty = M_\infty^2 - N_\infty$ and both M_∞^2 and N_∞ are integrable, we conclude that A_∞ is integrable.

We have now shown that A is almost surely increasing, that $\Delta A = (\Delta M)^2$ almost surely and that $M^2 - A$ is in $\mathcal{M}^2$. Now let F be the null set where A is not increasing and put $[M] = A1_{F^c}$. As we have assumed that all null sets are in $\mathcal{F}_t$ for $t \geq 0$, $[M]$ is adapted as A is adapted. Furthermore, $[M]$ is càdlàg, increasing and $[M]_\infty$ exists and is integrable, so $[M] \in \mathcal{A}^i$. Also, $\Delta[M] = (\Delta M)^2$ almost surely. As $M^2 - [M] = N + A1_F$, where $A1_F$ is evanescent and càdlàg and therefore in $\mathcal{M}^2$, the theorem is proven. □

Let $M \in \mathcal{M}^b$. As the process $[M]$ constructed in Theorem 1.3.6 satisfies $\Delta[M] = (\Delta M)^2$ almost surely, we obtain $0 \leq \sum_{0<t}(\Delta M_t)^2 \leq [M]_\infty$ almost surely. As $[M]_\infty$ is integrable, it is almost surely finite. Therefore, a nontrivial corollary of Theorem 1.3.6 is that almost surely, $\sum_{0<t}(\Delta M_t)^2$ is finite. In Chapter 3, we will show that this result in fact extends to all $M \in \mathcal{M}^2$.

1.4 Finite variation processes and integration

In this section, we prove a few results on the properties of processes with finite variation. We begin by introducing some additional notation. In the previous section, we defined $\mathcal{A}^i$ as the space of increasing and integrable càdlàg adapted stochastic processes with initial value zero. We now define $\mathcal{A}$ as the space of increasing càdlàg adapted stochastic processes with initial value zero, and we define $\mathcal{V}$ as the space of càdlàg adapted stochastic processes with initial value zero and paths of finite variation. Note that for a process in $\mathcal{V}$, the variation process is in $\mathcal{A}$. We say a process in $\mathcal{V}$ is integrable if the variation process is in $\mathcal{A}^i$, and denote this subspace of $\mathcal{V}$ by $\mathcal{V}^i$. We then have the inclusions $\mathcal{A}^i \subseteq \mathcal{A}$, $\mathcal{V}^i \subseteq \mathcal{V}$, $\mathcal{A}^i \subseteq \mathcal{V}^i$ and $\mathcal{A} \subseteq \mathcal{V}$.

We begin by proving a lemma which shows that elements of $\mathcal{V}$ can be decomposed in a measurable way, meaning that we may write elements of $\mathcal{V}$ as differences of elements of $\mathcal{A}$.

Lemma 1.4.1. *Let $A \in \mathcal{V}$. There exists processes $A^+, A^- \in \mathcal{A}$ such that $A = A^+ - A^-$. An explicit such decomposition is given by putting $A^+ = \frac{1}{2}(V_A + A)$ and $A^- = \frac{1}{2}(V_A - A)$.*

Proof. With A^+ and A^- defined as in the statement of the lemma, it is immediate that

$A = A^+ - A^-$. Furthermore, by Lemma 1.2.9, the processes A^+ and A^- are both càdlàg and adapted, and by Theorem A.2.9, A^+ and A^- are both increasing. This proves the result. □

Next, recall from Section A.2 that there is a bijective correspondence between mappings of finite variation and pairs of nonnegative, singular measures, so that mappings of finite variation may be used as integrators. For mappings with bounded variation, the two singular measures may be subtracted to obtain a signed measure, so that for such mappings, we obtain a correspondence with signed measures instead of paris of nonnegative, singular measures. We next seek to show that under certain measurability requirements for the integrand, we may construct the integral of a stochastic process with respect to elements of $\mathcal{V}$ in a measurable manner. This result will be important in Chapter 3. Recall from Section A.1 that a P-integrable $(\Omega, \mathcal{F})$ kernel on a countably generated measurable space $(E, \mathcal{E})$ is a family $(\nu_\omega)_{\omega \in \Omega}$ of signed measures on $(E, \mathcal{E})$ such that $\omega \mapsto \nu_\omega(A)$ is $\mathcal{F}$ measurable for all $A \in \mathcal{E}$ and such that $\int_\Omega |\nu_\omega|(E) \, \mathrm{d}P(\omega)$ is finite. Note also that as $(\mathbb{R}_+, \mathcal{B}_+)$ is generated by the open intervals with rational endpoints, $(\mathbb{R}_+, \mathcal{B}_+)$ is countably generated.

Lemma 1.4.2. *Let $A \in \mathcal{V}^i$ and assume that $(V_A)_\infty(\omega)$ is finite for all ω. For each ω, let $\nu_A(\omega)$ be the signed measure on $(\mathbb{R}_+, \mathcal{B}_+)$ induced by $A(\omega)$. The family $(\nu_A(\omega))_{\omega \in \Omega}$ is a P-integrable $(\Omega, \mathcal{F})$ kernel on $(\mathbb{R}_+, \mathcal{B}_+)$, and the restricted family $(\nu_A(\omega)_{|[0,t]})_{\omega \in \Omega}$ is a P-integrable $(\Omega, \mathcal{F}_t)$ kernel on $([0,t], \mathcal{B}_t)$.*

Proof. We first show the result on the family of restricted measures. Fix $t \geq 0$. The result is trivial for t equal to zero, so we may assume that t is positive. Let $\nu_A^t(\omega)$ be the restriction of $\nu_A(\omega)$ to $([0,t], \mathcal{B}_t)$. We first show that for any $B \in \mathcal{B}_t$, the mapping $\omega \mapsto \nu_A^t(\omega)(B)$ is $\mathcal{F}_t$ measurable. The family of $B \in \mathcal{B}_t$ for which this holds is a Dynkin class, and it will therefore suffice to show the claim for intervals of the type $(a,b]$ for $0 \leq a \leq b \leq t$. Let $0 \leq a \leq b \leq t$ be given. Then $\nu_A^t(\omega)((a,b]) = A_b(\omega) - A_a(\omega)$, and by the adaptedness of A, this is $\mathcal{F}_t$ measurable. Finally, also note that $\int_\Omega |\nu_A^t(\omega)|([0,t]) \, \mathrm{d}P(\omega) = E(V_A)_t$, which is finite by our assumptions. We conclude that $(\nu_A(\omega)_{|[0,t]})_{\omega \in \Omega}$ is a P-integrable $(\Omega, \mathcal{F}_t)$ kernel on $([0,t], \mathcal{B}_t)$.

Now consider the unrestricted case. Let $B \in \mathcal{B}_+$. Then $\nu_A(\omega)(B) = \lim_t \nu_A^t(\omega)(B)$ and $|\nu_A(\omega)|(B) = \lim_t |\nu_A^t(\omega)|(B)$. By what we already have shown, we find that $\omega \mapsto \nu_A(\omega)(B)$ is $\mathcal{F}$ measurable for all $B \in \mathcal{B}_+$. As $\int_\Omega |\nu_A(\omega)|(\mathbb{R}_+) \, \mathrm{d}P(\omega) = E(V_A)_\infty$, we conclude that $(\nu_A(\omega))_{\omega \in \Omega}$ is a P-integrable $(\Omega, \mathcal{F})$ kernel on $(\mathbb{R}_+, \mathcal{B}_+)$. □

Theorem 1.4.3. *Let $A \in \mathcal{V}$ and assume that H is progressive and that almost surely, H is integrable with respect to A. There is a process $H \cdot A \in \mathcal{V}$, unique up to indistinguishability,* *such that almost surely, $(H \cdot A)_t$ is the Lebesgue integral of H with respect to A over $[0,t]$ for all $t \geq 0$. If H is nonnegative and $A \in \mathcal{A}$, then $H \cdot A \in \mathcal{A}$.*

Proof. First note that as the requirements on $H \cdot A$ define the process pathwisely almost surely, it is immediate that $H \cdot A$ is unique up to indistinguishability. As for existence, we prove the result in three steps, first considering bounded A in $\mathcal{A}$, then general A in $\mathcal{A}$ and finally the case where we merely assume $A \in \mathcal{V}$.

Step 1. The case $A \in \mathcal{A}$, A bounded. First assume that $A \in \mathcal{A}$ and that A is bounded. Let F be the null set such that when $\omega \in F$, $H(\omega)$ is not integrable with respect to $A(\omega)$. By our assumptions on the filtration, $F \in \mathcal{F}_t$ for all $t \geq 0$, in particular we obtain that $\{(s,\omega) \in [0,t] \times \Omega \mid 1_F(\omega) = 1\} = [0,t] \times F \in \mathcal{B}_t \otimes \mathcal{F}_t$, and so the process $(t,\omega) \mapsto 1_F(\omega)$ is progressive. Therefore, the process $(t,\omega) \mapsto 1_{F^c}(\omega)$ is progressive as well. Thus, defining $K = H1_{F^c}$, K is progressive, and $K(\omega)$ is integrable with respect to $A(\omega)$ for all ω. We may then define a process Y by putting $Y_t(\omega) = \int_0^t K_s(\omega)\,\mathrm{d}A(\omega)_s$. We claim that Y satisfies the properties required of the process $H \cdot A$ in the statement of the lemma. Clearly, Y_t is almost surely the Lebesgue integral of H with respect to A over $[0,t]$ for all $t \geq 0$, it remains to prove $Y \in \mathcal{V}$. As Y is a pathwise Lebesgue integral with respect to a nonnegative measure, Y has finite variation. We would like to prove that Y is càdlàg. As Y is zero on F, it suffices to show that Y is càdlàg on F^c. Let $\omega \in F^c$ and let $t \geq 0$. For $h \geq 0$, we obtain by Lemma A.2.12 that

$$|Y_{t+h}(\omega) - Y_t(\omega)| = \left| \int_t^{t+h} H_s(\omega)\,\mathrm{d}A(\omega)_s \right| \leq \int_t^{t+h} |H_s(\omega)|\,|\,\mathrm{d}A(\omega)|_s.$$

We may then apply the dominated convergence theorem with the dominating function given by $s \mapsto |H_s(\omega)| 1_{(t,t+\varepsilon]}(s)$ for some $\varepsilon > 0$ to obtain

$$\limsup_{h \to 0} |Y_{t+h}(\omega) - Y_t(\omega)| \leq \lim_{h \to 0} \int_t^{t+h} |H_s(\omega)|\,|\,\mathrm{d}A(\omega)|_s = 0,$$

showing that $Y(\omega)$ is right-continuous at t. As Y has paths of finite variation, it is immediate that Y has left limits. This proves that Y is càdlàg. Furthermore, by construction, Y has initial value zero. Therefore, it only remains to prove that Y is adapted, meaning that Y_t is $\mathcal{F}_t$ measurable for all $t \geq 0$.

This is clearly the case for t equal to zero, therefore, assume that $t > 0$. Let $\nu_A^t(\omega)$ be the restriction to $\mathcal{B}_t$ of the nonnegative measure induced by $A(\omega)$ according to Theorem A.2.9. By Lemma 1.4.2, $(\nu_A(\omega)_{|[0,t]})_{\omega \in \Omega}$ is a P-integrable $(\Omega, \mathcal{F}_t)$ kernel on $([0,t], \mathcal{B}_t)$. Now, as K is progressive, the restriction of K to $[0,t] \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable. Theorem A.1.17 then yields that the integral $\int_0^t K_s(\omega)\,\mathrm{d}A(\omega)_s$ is $\mathcal{F}_t$ measurable, proving that Y_t is adapted. We conclude that $Y \in \mathcal{V}$.

Step 2. The case $A \in \mathcal{A}$. We now consider the case where $A \in \mathcal{A}$. Define a sequence (T_n) of positive stopping times by putting $T_n = \inf\{t \geq 0 \mid |A_t| > n\}$. As A has càdlàg paths and has initial value zero, the sequence (T_n) is positive and increases to infinity. For $0 \leq t < T_n$, it holds that $|A_t| \leq n$. Define A^{T_n-} by putting $A_t^{T_n-} = A_t$ when $0 \leq t < T_n$ and $A_t^{T_n-} = A_{T_n-}$ otherwise. A^{T_n-} is then a bounded element of $\mathcal{A}$. By what was already shown, there exists a process $H \cdot A^{T_n-}$ in $\mathcal{V}$ such that almost surely, $(H \cdot A^{T_n-})_t$ is the integral of H with respect to A^{T_n-} over $[0,t]$ for all $t \geq 0$. Let F be the null set such that on F^c, T_n converges to infinity, H is integrable with respect to A and $(H \cdot A^{T_n-})_t$ is the integral of H with respect to A^{T_n-} over $[0,t]$ for all $t \geq 0$ and all n. As before, we put $K = H1_{F^c}$ and conclude that K is progressive, and defining $Y_t = \int_0^t K_s\,\mathrm{d}A_s$, we find that Y_t is almost surely the Lebesgue integral of H with respect to A over $[0,t]$ for all $t \geq 0$. Furthermore, whenever $\omega \in F^c$, we have

$$Y_t = \int_0^t H_s\,\mathrm{d}A_s = \lim_{n\to\infty} \int_0^t 1_{[0,T_n)}(s) H_s\,\mathrm{d}A_s = \lim_{n\to\infty} \int_0^t H_s\,\mathrm{d}A_s^{T_n-} = \lim_{n\to\infty} (H \cdot A^{T_n-})_t,$$

where the second equality follows from the dominated convergence theorem, as H is assumed to be integrable with respect to A on F^c. In particular, Y_t is the almost sure limit of a sequence of $\mathcal{F}_t$ measurable variables. Therefore, Y_t is itself $\mathcal{F}_t$ measurable. Also, it follows that Y is càdlàg with paths of finite variation and has initial value zero. We conclude that $Y \in \mathcal{V}$ and so Y satisfies the requirements of the process $H \cdot A$ in the theorem.

Step 3. The case $A \in \mathcal{V}$. Finally, assume that $A \in \mathcal{V}$. Recalling Theorem A.2.9, we know that by putting $A_t^+ = \frac{1}{2}((V_A)_t + A_t)$ and $A_t^- = \frac{1}{2}((V_A)_t + A_t)$, H is almost surely integrable with respect to A^+ and A^-. By Lemma 1.4.1, A^+ and A^- are in $\mathcal{A}$ and $A = A^+ - A^-$.

By what was already shown, there are processes $H \cdot A^+$ and $H \cdot A^-$ in $\mathcal{V}$ such that almost surely, these processes at time t are the Lebesgue integrals of H with respect to A^+ and A^- over $[0,t]$ for all $t \geq 0$. The process $H \cdot A = H \cdot A^+ - H \cdot A^-$ then satisfies the requirements of the theorem. $\square$

Theorem 1.4.3 shows that given a progressively measurable process H and $A \in \mathcal{V}$ such that H is almost surely integrable with respect to A, which as in Section A.2 means that H is integrable with respect to the nonnegative singular measures induced by the increasing and decreasing parts of A on $[0,t]$ for any $t \geq 0$, we may define the integral pathwisely in such a manner as to obtain a process $H \cdot A \in \mathcal{V}$, where it holds for almost all ω that for any $t \geq 0$,

$(H \cdot A)_t(\omega) = \int_0^t H_s(\omega)\,\mathrm{d}A(\omega)_s$. The stochastic integral of H with respect to A thus becomes another stochastic process.

1.5 Exercises

Exercise 1.5.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability triple and let $(\mathcal{G}_n)_{n\geq 0}$ be a discrete-time filtration on $(\Omega, \mathcal{F}, P)$. For $n \in \mathbb{N}_0$ and $n \leq t < n+1$, define $\mathcal{F}_t = \mathcal{G}_n$. Show that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous.*

Exercise 1.5.2. *Let X be an adapted stochastic process. Assume that for all $\varepsilon > 0$, X is progressive with respect to the filtration $(\mathcal{F}_{t+\varepsilon})_{t\geq 0}$. Show that X is progressive with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

Exercise 1.5.3. *Let X be continuous and adapted, and let F be a closed set. Define a mapping T by putting $T = \inf\{t \geq 0 \mid X_t \in F\}$. Show that T is a stopping time.*

Exercise 1.5.4. *Let X be càdlàg and adapted, and let F be a closed set. Define a mapping T by putting $T = \inf\{t \geq 0 \mid X_t \in F \text{ or } X_{t-} \in F\}$. Show that T is a stopping time.*

Exercise 1.5.5. *Let X be a continuous and adapted stochastic process, and let $a \in \mathbb{R}$. Put $T = \inf\{t \geq 0 \mid X_t = a\}$. Prove that T is a stopping time and that $X_T = a$ whenever $T < \infty$.*

Exercise 1.5.6. *Let S and T be two stopping times. Show that $\mathcal{F}_{S\vee T} = \sigma(\mathcal{F}_S, \mathcal{F}_T)$.*

Exercise 1.5.7. *Let (T_n) be a decreasing sequence of stopping times with limit T. Show that T is a stopping time and that $\mathcal{F}_T = \cap_{n=1}^\infty \mathcal{F}_{T_n}$.*

Exercise 1.5.8. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Put $M_t = W_t^2 - t$. Show that M is not uniformly integrable.*

Exercise 1.5.9. *Let M be a càdlàg adapted process with initial value zero and assume that M_t is almost surely convergent. Show that $M \in \mathcal{M}^u$ if and only if M_T is integrable with $EM_T = 0$ for all stopping times T which take at most two values in $[0, \infty]$.*

Exercise 1.5.10. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and let $\alpha \in \mathbb{R}$. Show that the process M^α defined by $M_t^\alpha = \exp(\alpha W_t - \frac{1}{2}\alpha^2 t)$ is a martingale. Let $a \in \mathbb{R}$ and define $T = \inf\{t \geq 0 \mid W_t = a\}$. Show that for any $\beta \geq 0$, $E\exp(-\beta T) = \exp(-|a|\sqrt{2\beta})$.*

Exercise 1.5.11. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Show by direct calculation that the processes $W_t^3 - 3tW_t$ and $W_t^4 - 6tW_t^2 + 3t^2$ are in $\mathbf{c}\mathcal{M}$.*

Exercise 1.5.12. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and define T by putting $T = \inf\{t \geq 0 \mid W_t \geq a + bt\}$. Show that T is a stopping time and that for $a > 0$ and $b > 0$, it holds that $P(T < \infty) = \exp(-2ab)$.*

Exercise 1.5.13. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Let $a > 0$ and define $T = \inf\{t \geq 0 \mid W_t^2 \geq a(1-t)\}$. Show that T is a stopping time. Find ET and ET^2.*

Exercise 1.5.14. *Let N be an $(\mathcal{F}_t)$ Poisson process. Define $M_t = N_t^2 - 2tN_t + t^2 - t$. Show that M is a martingale.*

Exercise 1.5.15. *Let N be an $\mathcal{F}_t$ Poisson process and let $\alpha \in \mathbb{R}$. Show that the process M^α defined by $M_t^\alpha = \exp(\alpha N_t - (e^\alpha - 1)t)$ is a martingale.*

Chapter 2

Predictability and stopping times

In this chapter, we introduce the predictable σ-algebra and the related concept of predictable stopping times, and we also introduce two further subclasses of stopping times, namely accessible and totally inaccessible stopping times. Using the results of this chapter, we will in Chapter 3 see that martingales have a special interplay with predictable stopping times and the predictable σ-algebra, and this interplay is the reason for the importance of the predictable σ-algebra and predictable stopping times.

The main result of this chapter is a characterisation of measurability with respect to the predictable σ-algebra for càdlàg adapted processes in terms of the behaviour at jump times. This result will be used repeatedly in Chapter 3.

The structure of the chapter is as follows. In Section 2.1, we introduce the predictable σ-algebra Σ^p and predictable stopping times. We identify generators for Σ^p, and prove the elementary stability properties of predictable stopping times. The main result is Theorem 2.1.12, which gives a precise characterisation of predictable stopping times in terms of their graphs.

Next, in Section 2.2, we introduce the σ-algebra $\mathcal{F}_{T-}$ of events strictly prior to T and consider its elementary properties. Using this σ-algebra, we are able to prove some miscellaneous results on predictable stochastic processes, results which will be used to demonstrate the main results of the following section. In the final section, Section 2.3, we show that all stopping times may be decomposed into two parts, named the accessible and the totally inaccessible parts. We argue that the jumps times of any càdlàg process may be covered by a countable family of such stopping times. Finally, in Theorem 2.3.9, we characterize predictability of adapted càdlàg processes in terms of their behaviour at jump times.

2.1 The predictable σ-algebra

In this section, we introduce and investigate the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$, as well as predictable stopping times, which are stopping times interacting with the predictable σ-algebra in a particular manner. We begin by defining the predictable σ-algebra and identifying a few generators for the σ-algebra. Recall that a mapping $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be càglàd if it is left-continuous on $(0, \infty)$ with right limits on $\mathbb{R}_+$. Also, a process is said to be càglàd if its sample paths are càglàd.

Definition 2.1.1. *The predictable σ-algebra Σ^p is the σ-algebra on $\mathbb{R}_+ \times \Omega$ generated by the adapted càglàd processes.*

By $\mathcal{T}$, we denote the family of all stopping times. In the following lemmas, we work towards identifying some generators for Σ^p.

Lemma 2.1.2. *Let X be left-continuous. Define a process X^n by putting*

$$X^n_t = 1_{\{0\}}(t)X_0(\omega) + \sum_{k=0}^{\infty} X_{k2^{-n}}(\omega)1_{(k2^{-n},(k+1)2^{-n}]}(t).$$

Then X^n converges pointwise to X.

Proof. Fix $\omega \in \Omega$. It is immediate that $X^n_0(\omega)$ converges to $X_0(\omega)$, so it suffices to show that $X^n_t(\omega)$ converges to $X_t(\omega)$ for $t > 0$. Fix such a $t > 0$. Take $\varepsilon > 0$ and pick $\delta \leq t$ such that $|X_t(\omega) - X_s(\omega)| \leq \varepsilon$ when $s \in [t - \delta, t]$. Take n so large that $2^{-n} \leq \delta$. There is k satisfying $k2^{-n} < t \leq (k+1)2^{-n}$ such that $X^n_t(\omega) = X_{k2^{-n}}(\omega)$. In particular, we obtain that $k2^{-n} \in [t - 2^{-n}, t] \subseteq [t - \delta, t]$, so $|X_t(\omega) - X^n_t(\omega)| \leq \varepsilon$. This shows that $X^n_t(\omega)$ converges to $X_t(\omega)$ and thus proves the lemma. ∎

Lemma 2.1.3. *Σ^p is also generated by the set families $\{\{0\}\times F \mid F\in\mathcal{F}_0\}\cup\{\rrbracket T,\infty\llbracket | T\in\mathcal{T}\}$ and $\{\{0\}\times F \mid F\in\mathcal{F}_0\}\cup\{\rrbracket S,T\rrbracket | S,T\in\mathcal{T}\}$.*

Proof. Let S and T be stopping times. Noting that $\rrbracket T,\infty\llbracket = \cup_{n=1}^\infty \rrbracket T,T+n\rrbracket$, where $T+n$ is a stopping time by Lemma 1.1.9, and $\rrbracket S,T\rrbracket = \rrbracket S,\infty\llbracket \setminus \rrbracket T,\infty\llbracket$, we obtain that the two families generate the same σ-algebra. It will therefore suffice to show that Σ^p is generated by, say, $\{\{0\}\times F \mid F\in\mathcal{F}_0\}\cup\{\rrbracket S,T\rrbracket | S,T\in\mathcal{T}\}$. Let $\mathcal{H}$ be the σ-algebra generated by this set family, we wish to show $\Sigma^p=\mathcal{H}$.

We first argue that $\mathcal{H}\subseteq\Sigma^p$, and to this end, we show that Σ^p contains a generator for $\mathcal{H}$. To this end, first consider $F\in\mathcal{F}_0$. Define $X_t(\omega)=1_F(\omega)1_{\{0\}}(t)$. Then X is càglàd and adapted, so X is Σ^p measurable. Therefore, $\{0\}\times F\in\Sigma^p$. Next, consider two stopping times S and T, and put $X=1_{\rrbracket S,T\rrbracket}$. Then X is càglàd, and as $X_t=1_{(S<t\le T)}$, Lemma 1.1.9 shows that X is adapted. Therefore, X is Σ^p measurable, implying that $\rrbracket S,T\rrbracket\in\Sigma^p$. Thus, Σ^p contains a generator for $\mathcal{H}$, so $\mathcal{H}\subseteq\Sigma^p$.

It remains to prove the other inclusion. To do so, it will suffice to prove that any càglàd adapted process is $\mathcal{H}$-measurable. To this end, we first consider some simple càglàd adapted processes. First off, note that $1_F(\omega)1_{\{0\}}(t)$ is $\mathcal{H}$ measurable when $F\in\mathcal{F}_0$. Since any bounded $\mathcal{F}_0$-measurable variable can be approximated by simple functions, it follows that $Z(\omega)1_{\{0\}}(t)$ is $\mathcal{H}$ measurable for any bounded $\mathcal{F}_0$ measurable Z. Since any $\mathcal{F}_0$ measurable variable can be approximated by bounded $\mathcal{F}_0$ measurable variables, we finally conclude that any $Z(\omega)1_{\{0\}}(t)$ is $\mathcal{H}$ measurable for any Z which is $\mathcal{F}_0$ measurable. Next, consider a process of the type $1_H(\omega)1_{(s,u]}(t)$, where $H\in\mathcal{F}_s$. Using the notation of Lemma 1.1.9, consider the stopping times $S=s_H$ and $T=u_H$, meaning that S is equal to s on H and infinity otherwise. We then have

$$\begin{aligned}\{(t,\omega)\in\mathbb{R}_+\times\Omega \mid 1_H(\omega)1_{(s,u]}(t)=1\} &= \{(t,\omega)\in\mathbb{R}_+\times\Omega \mid s<t\le u,\omega\in H\}\\ &= \{(t,\omega)\in\mathbb{R}_+\times\Omega \mid S(\omega)<t\le T(\omega)\}\\ &= \rrbracket S,T\rrbracket,\end{aligned}$$

so $1_H(\omega)1_{(s,t]}(t)$ is $\mathcal{H}$-measurable. By approximation arguments as for the previous class of processes, $Z1_{(s,u]}$ is $\mathcal{H}$-measurable whenever $Z\in\mathcal{F}_s$. Finally, let X be any càglàd adapted process. Define $X^n_t=1_{\{0\}}(t)X_0(\omega)+1_{(0,\infty)}(t)\sum_{k=0}^\infty X_{k2^{-n}}(\omega)1_{(k2^{-n},(k+1)2^{-n}]}(t)$. Since X is adapted, the above is an infinite sum of processes which all are $\mathcal{H}$ measurable according to what already has been shown. By Lemma 2.1.2, X^n converges pointwise to X. Thus, as X^n is $\mathcal{H}$ measurable, X is $\mathcal{H}$ measurable. Ergo, $\Sigma^p\subseteq\mathcal{H}$, as desired. $\square$

The above result, modulo considerations about the timepoint zero, shows that Σ^p is generated by $]\!]T,\infty[\![$ for $T \in \mathcal{T}$. By analogy, $\mathbb{B}_+$ is generated by the family of sets (t,∞) for $t \geq 0$, but also by $[t,\infty)$ for $t \geq 0$. It is therefore natural to ask whether $[\![T,\infty[\![$ also would generate Σ^p. The immediate idea for the proof of such a claim would be to make the approximation $]\!]T,\infty[\![= \cap_{n=1}^\infty [\![T-\frac{1}{n},\infty[\![$. However, $T-\frac{1}{n}$ may not be a stopping time, and therefore this avenue of proof fails. Motivated by these remarks, we now introduce the notion of predictable stopping times.

Definition 2.1.4. *Let T be a stopping time. We say that T is predictable if there is an sequence of stopping times (T_n) increasing pointwise to T such that whenever $T(\omega) > 0$ it holds that $T_n(\omega) < T(\omega)$ for all n. We say that (T_n) is an announcing sequence for T. The set of predictable stopping times is denoted by $\mathcal{T}_p$.*

The following lemma yields a few stability properties of predictable stopping times.

Lemma 2.1.5. *If T is a predictable stopping time and and $F \in \mathcal{F}_0$, then T_F is a predictable stopping time. If S and T are predictable stopping times, so are $S \wedge T$, $S \vee T$ and $S+T$. Finally, for any constant $c \in [0,\infty]$ and $F \in \mathcal{F}_0$, c_F is a predictable stopping time.*

Proof. We already know from Lemma 1.1.9 that all the variables mentioned are stopping times, so it will suffice to prove predictability.

Consider first the case where T is a predictable stopping time and $F \in \mathcal{F}_0$. We want to show that T_F is a predictable stopping time. Let (T_n) be an announcing sequence for T, and put $S_n = n \wedge (T_n)_F$. We claim that (S_n) is an announcing sequence for T_F. To show this, first note that as $F \in \mathcal{F}_0 \subseteq \mathcal{F}_{T_n}$, we find that $(T_n)_F$ and thus S_n are stopping times by Lemma 1.1.9. It is immediate that (S_n) is increasing. On F, we have $S_n = n \wedge T_n$ and $T_F = T$, and on F^c, we have $S_n = n$ and $T_F = \infty$. Therefore, S_n increases to T_F, and if $T_F > 0$, then $S_n < T_F$. This proves the result. As an immediate corollary, we also obtain that c_F is a predictable stopping time for any $c \in [0,\infty]$ and $F \in \mathcal{F}_0$.

Next, let S and T be two predictable stopping times, and let (S_n) and (T_n) be announcing sequences for S and T, respectively.

Considering $S \wedge T$, we will argue that $(S_n \wedge T_n)$ is an announcing sequence for $S \wedge T$. It is immediate that $S_n \wedge T_n$ increases to $S \wedge T$, so it suffices to argue that $S_n \wedge T_n < S \wedge T$ whenever $S \wedge T > 0$. However, when $S \wedge T > 0$, we have that both S and T are positive. Therefore, both S_n and T_n are strictly less than S and T, respectively, and so $S_n \wedge T_n < S \wedge T$, as desired. Thus, $(S_n \wedge T_n)$ is an announcing sequence for $S \wedge T$ and so $S \wedge T$ is predictable.

Regarding $S \vee T$, we claim that $(S_n \vee T_n)$ is an announcing sequence for $S \vee T$. As in the previous case, $S_n \vee T_n$ increases to $S \vee T$, so it suffices to argue that $S_n \vee T_n < S \vee T$ whenever $S \vee T > 0$. In this case, $S \vee T > 0$ implies that either $S > 0$ or $T > 0$. If $S > 0$ and $T \geq S$, we obtain $S \vee T = T$, $T > 0$ and thus $S_n < S \leq T$ and $T_n < T$, yielding $S_n \vee T_n < S \vee T$. If $S > 0$ and $T < S$, we obtain $S \vee T = S$ and so $S_n < S$ and $T_n \leq T < S$, yielding $S_n \vee T_n < S \vee T$ in this case as well. The cases $T > 0$, $S \geq T$ and $T > 0$, $S < T$ may be handled similarly, in total demonstrating that whenever $S \vee T > 0$, we have $S_n \vee T_n < S \vee T$. Consequently, $(S_n \vee T_n)$ is an announcing sequence for $S \vee T$, so $S \vee T$ is predictable.

Finally, we consider $S + T$. Here, we wish to show that $(S_n + T_n)$ is an announcing sequence for $S + T$. It is immediate that $S_n + T_n$ increases to $S + T$. If $S + T$ is positive, at least one of S and T is positive. If S is positive, $S_n < S$ and thus $S_n + T_n < S + T$. Similarly, we obtain $S_n + T_n < S + T$ when T is positive. We conclude that $(S_n + T_n)$ is an announcing sequence for $S + T$ and thus $S + T$ is predictable. $\square$

Lemma 2.1.6. *Σ^p is generated by $\{[\![T, \infty[\![\,|\, T \in \mathcal{T}_p\}$ and by $\{[\![S, T[\![\,|\, S, T \in \mathcal{T}_p\}$.*

Proof. Since the constant infinity is a predictable stopping time by by Lemma 2.1.5, we immediately obtain $\sigma(\{[\![T, \infty[\![\,|\, T \in \mathcal{T}_p\}) \subseteq \sigma(\{[\![S, T[\![\,|\, S, T \in \mathcal{T}_p\})$. The other inclusion follows from the relation $[\![S, T[\![= [\![S, \infty[\![\setminus [\![T, \infty[\![$, where $S, T \in \mathcal{T}_p$. Therefore, the two families generate the same σ-algebra, and it will suffice to prove that this σ-algebra is in fact Σ^p.

We will prove that Σ^p is generated by $\{[\![T, \infty[\![\,|\, T \in \mathcal{T}_p\}$. Let $\mathcal{H}$ be the σ-algebra generated by $\{[\![T, \infty[\![\,|\, T \in \mathcal{T}_p\}$. We need to show $\Sigma^p = \mathcal{H}$. Assume first that T is predictable and let (T_n) be an announcing sequence. Then $[\![T, \infty[\![= \{0\} \times (T = 0) \cup \cap_{n=1}^\infty]\!]T_n, \infty]\!]$. Since the first part of the union is in $\{\{0\} \times F \mid F \in \mathcal{F}_0\}$, we can use Theorem 2.1.3 to conclude that the above is in Σ^p. Therefore, $\mathcal{H} \subseteq \Sigma^p$. To prove the other inclusion, we recall from Theorem 2.1.3 that Σ^p is generated by $\{\{0\} \times F \mid F \in \mathcal{F}_0\}$ and $\{]\!]T, \infty[\![\,|\, T \in \mathcal{T}\}$. First note that for any $F \in \mathcal{F}_0$, we have

$$
\begin{aligned}
\{0\} \times F &= \{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid \omega \in F\} \setminus \{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid t > 0, \omega \in F\} \\
&= \{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid \omega \in F\} \setminus \cup_{n=1}^\infty \{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid t \geq \tfrac{1}{n}, \omega \in F\} \\
&= [\![0_F, \infty[\![\setminus \cup_{n=1}^\infty [\![(\tfrac{1}{n})_F, \infty[\![.
\end{aligned}
$$

Now, by Lemma 2.1.5, 0_F and $(\frac{1}{n})_F$ are both predictable stopping times. Therefore, the above is in $\mathcal{H}$. This proves $\{\{0\} \times F \mid F \in \mathcal{F}_0\} \subseteq \mathcal{H}$. Next, letting T be any stopping time, we have $]\!]T, \infty[\![= \cup_{n=1}^\infty [\![T + \frac{1}{n}, \infty[\![$. The stopping time $T + \frac{1}{n}$ is predictable with announcing sequence $T + \max\{\frac{1}{n} - \frac{1}{k}, 0\}$ for $k \geq 1$, and therefore also the above is in $\mathcal{H}$. This proves $\{]\!]T, \infty[\![\,|\, T \in \mathcal{T}\} \subseteq \mathcal{H}$ and therefore finally $\Sigma^p \subseteq \mathcal{H}$. $\square$

Lemma 2.1.6 shows that predictable stopping times have a natural interplay with the predictable σ-algebra. Also, while the generators obtained in Lemma 2.1.3 contained the unwieldy $\{\{0\} \times F \mid F \in \mathcal{F}_0\}$ term, the generators of Lemma 2.1.6 are easier to work with.

In the remainder of the section, we will prove some results on predictable stopping times. We will show some further stability properties of the class of predictable stopping times, and at the end of the section, in Theorem 2.1.12, we will show that a stopping time T is predictable if and only if its graph $[\![T]\!]$ is a predictable set in the sense of being an element of Σ^p.

Lemma 2.1.7. *Let T be a stopping time and let T_n be a sequence of stopping times such that almost surely, T_n increases to T, and almost surely, when $T > 0$, $T_n < T$ for all n. Then T is predictable.*

Proof. The nontrivial part of the lemma is that the properties of the sequence T_n only hold almost surely. Let F be the almost sure set such that for $\omega \in F$, $T_n(\omega)$ increases to T, and if $T(\omega) > 0$, $T_n(\omega) < T(\omega)$. Define $S_n = (T_n)_F \wedge (\max\{T - \frac{1}{n}, 0\})_{F^c}$. We claim that (S_n) is an announcing sequence for T. To this end, first note that as F^c is a null set and we have assumed the usual conditions, both F and F^c are in $\mathcal{F}_0$. Therefore, $(T_n)_F$ is a stopping time by Lemma 1.1.9, and $(\max\{T - \frac{1}{n}, 0\})_{F^c}$ is a stopping time as it is almost surely zero. Thus, (S_n) is a sequence of stopping times. It is immediate that (S_n) is increasing. Also, for $\omega \in F$, we have that $\lim_{n\to\infty} S_n(\omega) = \lim_{n\to\infty} T_n(\omega) = T(\omega)$, and if $\omega \in F^c$, $\lim_{n\to\infty} S_n(\omega) = \lim_{n\to\infty} \max\{T - \frac{1}{n}, 0\} = T(\omega)$. Thus, S_n increases to T. Now assume $T(\omega) > 0$. If $\omega \in F$, we have $S_n(\omega) = T_n(\omega) < T(\omega)$, and if $\omega \in F^c$, we have $S_n(\omega) = \max\{T(\omega) - \frac{1}{n}, 0\} < T(\omega)$. Thus, $S_n < T$ whenever $T > 0$. We have now shown that (S_n) is an announcing sequence for T, so T is a predictable stopping time. $\square$

Lemma 2.1.8. *Let S and T be two nonnegative variables and assume that S and T are equal almost surely. If T is a stopping time, so is S. If T is predictable, so is S.*

Proof. We have

$$(S \leq t) = (S \leq t, S = T) \cup (S \leq t, S \neq T) = ((T \leq t) \cap (S = T)) \cup (S \leq t, S \neq T).$$

Now, $\mathcal{F}_t$ contains all P null sets, and therefore also all P almost sure sets. In particular, $(S = T) \in \mathcal{F}_t$ and $(S \leq t, S \neq T) \in \mathcal{F}_t$. Since T is a stopping time, $(T \leq t) \in \mathcal{F}_t$ and so S is a stopping time. And if T is predictable, Lemma 2.1.7 shows that S also is predictable. $\square$

Lemma 2.1.9. *Let (T_n) be a sequence of predictable stopping times. Then $\sup_n T_n$ is a predictable stopping time. If T_n is decreasing such that pointwisely, T_n is constant from some point onwards, then $\inf_n T_n$ is a predictable stopping time.*

Proof. As we know from Lemma 1.1.10 that $\sup_n T_n$ and $\inf_n T_n$ both are stopping times, we merely need to show predictability. Consider the case of the supremum. Put $T = \sup_n T_n$. As $T = \sup_n \max_{k\leq n} T_k$ and $\max_{k\leq n} T_k$ is a predictable stopping time by Lemma 2.1.5, it suffices to prove that $\sup_n T_n$ is a predictable stopping time under the extra condition that T_n increases. Let (T_n^k) be an announcing sequence for T_n. Define $S_n = \max_{i\leq n}\max_{k\leq n} T_i^k$. We claim that S_n announces T. It is immediate that S_n is an increasing sequence of stopping times, and as

$$\lim S_n = \sup_n \max_{i\leq n}\max_{k\leq n} T_i^k = \sup_n \sup_k T_n^k = \sup_n T_n = T,$$

S_n in fact increases to T. It remains to show that if $T > 0$, then $S_n < T$ for all n. To this end, first note that for any i and k, it holds that if $T_i^k > 0$, then $T_i > 0$ and thus $T_i^k < T_i \leq T$. Thus, whenever $T_i^k > 0$, then $T_i^k < T$. Now assume that $T > 0$ and consider some $n \geq 1$. If $S_n = 0$, we immediately have $S_n < T$. If $S_n > 0$, we have $S_n = T_i^k$ for some $i, k \leq n$, and thus $S_n < T$ by what was already shown. Thus, (S_n) announces T.

Next, assume that T_n decreases such that T_n pointwisely is constant from some point onwards and put $T = \inf_n T_n$, we want to show that T is predictable. For each n, let (T_n^k) be an announcing sequence for T_n. Define a metric d on $[0,\infty]$ by $d(x,y) = |e^{-x} - e^{-y}|$, with the convention that $e^{-x} = 0$ when x is infinite. We then have $\lim_n d(T_n, T) = 0$ pointwisely, and for each n, $\lim_k d(T_n^k, T_n) = 0$ pointwisely. In particular, $1_{(d(T_n^k,T_n)>\varepsilon)}$ converges pointwisely to zero. Applying the dominated convergence theorem, we then get for all n and $\varepsilon > 0$ that

$$\lim_k P(d(T_n^k, T_n) > \varepsilon) = \lim_k E1_{(d(T_n^k,T_n)>\varepsilon)} = E\lim_k 1_{(d(T_n^k,T_n)>\varepsilon)} = 0.$$

Now choose p_n such that $P(d(T_n^{p_n}, T_n) > \frac{1}{n}) \leq 2^{-n}$, and define $S_n = \inf_{k\geq n} T_k^{p_k}$. We claim that S_n almost surely announces T.

Since S_n is the infimum of a sequence of stopping times, it is clear that S_n is a stopping time as well. As the set over which the infimum is taken decreases with n, S_n increases with n, in particular the limit always exists. To show that S_n in fact increases to T almost surely, first note that by construction, $\sum_{n=1}^\infty P(d(T_n^{p_n}, T_n) > \frac{1}{n})$ is finite, and therefore $P(\cap_{n=1}^\infty \cup_{k=n}^\infty (d(T_k^{p_k}, T_k) > \frac{1}{k})) = 0$ by the Borel-Cantelli lemma. In particular, it holds for any $\varepsilon > 0$ that

$$P(\cap_{n=1}^\infty \cup_{k=n}^\infty (d(T_k^{p_k}, T_k) > \varepsilon)) \leq P(\cap_{n=1}^\infty \cup_{k=n}^\infty (d(T_k^{p_k}, T_k) > \tfrac{1}{k})) = 0.$$

As T_n is equal to T from some point onwards pointwise, we then obtain for any $\varepsilon > 0$ that

$$\begin{aligned} P(d(\lim_n S_n, T) > \varepsilon) &= P(\lim_n d(S_n, T) > \varepsilon) \leq P(\cap_{n=1}^\infty \cup_{k=n}^\infty (d(S_k, T) > \varepsilon)) \\ &\leq P(\cap_{n=1}^\infty \cup_{k=n}^\infty (d(T_k^{p_k}, T_k) > \varepsilon)) = 0. \end{aligned}$$

As ε was arbitrary, we conclude $P(d(\lim S_n, T) > 0) = 0$ and thus T is the almost sure limit of S_n. It remains that show that $S_n < T$ whenever $T > 0$. To this end, assume that $T(\omega) > 0$ and let $n \geq 1$, we want to show $S_n(\omega) < T(\omega)$. Let $N(\omega)$ be the first natural such that for any $k \geq N(\omega)$, $T_k(\omega) = T(\omega)$. As the infimum is smaller than each of the elements the infimum is taken over, we have $S_n(\omega) = \inf_{k\geq n} T_k^{p_k}(\omega) \leq T_{n\vee N(\omega)}^{p_{n\vee N(\omega)}}(\omega)$. As $T_{n\vee N(\omega)}(\omega) \geq T(\omega) > 0$, this implies $S_n(\omega) \leq T_{n\vee N(\omega)}^{p_{n\vee N(\omega)}}(\omega) < T_{n\vee N(\omega)}(\omega) = T(\omega)$, as desired. Lemma 2.1.7 now yields that T is predictable. □

Lemma 2.1.10. *Let S and T be stopping times. If S is predictable, $S_{(S\leq T)}$ is a predictable stopping time. If S and T are both predictable, $S_{(S<T)}$ is a predictable stopping time.*

Proof. First assume merely that S is a predictable stopping time, we wish to show that $S_{(S\leq T)}$ is a predictable stopping time. As $(S \leq T) \in \mathcal{F}_{S\wedge T} \subseteq \mathcal{F}_S$ by Lemma 1.1.11, Lemma 1.1.9, shows that $S_{(S\leq T)}$ is a stopping time. We need to prove that it is predictable. To this end, let (S_n) be an announcing sequence for S and define $U_n = n \wedge (S_n)_{(S_n\leq T)}$. We claim that (U_n) is an announcing sequence for $S_{(S\leq T)}$. To show this, first note that as S_n is increasing, the sequence of sets $(S_n \leq T)$ is decreasing, so $(S_n)_{(S_n\leq T)}$ is increasing and therefore, U_n is increasing. Furthermore,

$$\begin{aligned}
\sup_n U_n &= \sup_n n \wedge (S_n)_{(S_n\leq T)} = \sup_n (S_n)_{(S_n\leq T)} = \sup_n S_n 1_{(S_n\leq T)} + \infty 1_{(S_n\leq T)^c} \\
&= (\sup_n S_n) 1_{\cap_{n=1}^\infty (S_n\leq T)} + \infty 1_{\cup_{n=1}^\infty (S_n\leq T)^c} = S 1_{(S\leq T)} + \infty 1_{(S\leq T)^c} = S_{(S\leq T)},
\end{aligned}$$

so U_n increases to $S_{(S\leq T)}$. We need to prove that if $S_{(S\leq T)} > 0$, then $U_n < S_{(S\leq T)}$. To this end, note that on $(S \leq T)$, if $S > 0$, we have $S_n < S$ and so $U_n \leq (S_n)_{(S\leq T)} < S_{(S\leq T)}$. On $(S \leq T)^c$, it holds that U_n is finite while $S_{(S\leq T)}$ is infinite, so $U_n < S_{(S\leq T)}$ as well. This shows that $S_{(S\leq T)}$ is predictable when S is predictable.

Now assume that S and T are both predictable, we wish to show that $S_{(S<T)}$ is a predictable stopping time. Again by Lemma 1.1.11 and Lemma 1.1.9, we know that $S_{(S<T)}$ is a stopping time, so we merely need to show that it is predictable. To this end, let (T_n) be an announcing sequence for T and define $U_n = S_{(T>0,S\leq T_n)}$. We claim that (U_n) is a sequence of predictable stopping times decreasing to $S_{(S<T)}$ which pointwisely is constant from a point onwards, if we can prove this, Lemma 2.1.9 will show that $S_{(S<T)}$ is predictable.

We first show that U_n is predictable. To this end, define $V_n = S_{(S\leq T_n)}$ and note that $U_n = (V_n)_{(T>0)}$. As V_n is predictable from what we already have shown, Lemma 2.1.5 yields that U_n is predictable as well. We next show that (U_n) decreases to $S_{(S<T)}$ and pointwisely is constant from a point onwards. To see this, first note that as T_n is increasing, the sequence

of sets $(T > 0, S \leq T_n)$ is increasing as well. Therefore, (U_n) is decreasing, and

$$\begin{aligned} \inf_n U_n &= \inf_n S1_{(T>0,S\leq T_n)} + \infty 1_{(S\leq T_n)^c} \\ &= S1_{\cup_{n=1}^\infty (T>0,S\leq T_n)} + \infty 1_{\cap_{n=1}^\infty (T>0,S\leq T_n)^c} = S_{\cup_{n=1}^\infty (T>0,S\leq T_n)}. \end{aligned}$$

Now, on the set $\cup_{n=1}^\infty (T > 0, S \leq T_n)$, it holds that $T_n < T$ and so $S < T$. Conversely, if $S < T$, we have in particular $T > 0$ and there is n such that $S \leq T_n$. Thus, we conclude $\cup_{n=1}^\infty (T > 0, S \leq T_n) = (S < T)$, and so $\inf_n U_n = S_{(S<T)}$. Finally, as U_n only takes the values S and ∞ and the sequence of sets $(T > 0, S \leq T_n)$ is increasing, U_n is constant from a point onwards. Lemma 2.1.9 now shows that $S_{(S<T)}$ is predictable. $\square$

We now begin to work towards the final result of this section, namely the equivalence between being a predictable stopping time and having $[\![T]\!]$ predictable. To prove this result, we introduce the concept of the debut of a set. Let A be a subset of $\mathbb{R}_+ \times \Omega$. The debut D_A of A is the mapping from Ω to $[0, \infty]$ defined by $D_A(\omega) = \inf\{t \geq 0 \mid (t, \omega) \in A\}$.

Also, let $\mathbb{I}_p$ denote the family of finite unions of sets of the form $[\![S, T[\![$, where S and T are predictable stopping times with $S \leq T$. By Lemma A.1.7, $\mathbb{I}_p$ is an algebra generating the predictable σ-algebra. We let $(\mathbb{I}_p)_\delta$ denote the sets which may be obtained as countable intersections of elements of $\mathbb{I}_p$. By Lemma A.1.8, it holds for any bounded nonnegative measure μ on the predictable σ-algebra that all elements of the predictable σ-algebra can be approximated in μ-measure from the inside by elements of $(\mathbb{I}_p)_\delta$.

Lemma 2.1.11. *Consider some $A \in (\mathbb{I}_p)_\delta$. Then the debut D_A is a predictable stopping time, and for $\omega \in (D_A < \infty)$, it holds that $(D_A(\omega), \omega) \in A$.*

Proof. Let $A \in \mathbb{I}_p$ be given. Assume that $A = \cap_{n=1}^\infty A_n$, where (A_n) is a family of sets in $\mathbb{I}_p$. Specifically, assume that $A_n = \cup_{k=1}^{m_n} [\![S_{nk}, T_{nk}[\![$, where S_{nk} and T_{nk} are predictable stopping times with $S_{nk} \leq T_{nk}$.

We first argue that for $\omega \in (D_A < \infty)$, it holds that $(D_A(\omega), \omega) \in A$. To this end, assume that $\omega \in (D_A < \infty)$. Define $A(\omega) = \{t \geq 0 \mid (t, \omega) \in A\}$. There is a sequence $(\alpha_n(\omega))$ such that $\alpha_n(\omega) \in A(\omega)$ and $\alpha_n(\omega)$ converges downwards to $D_A(\omega)$. As $\alpha_n(\omega) \in A(\omega)$, we have $S_{nk}(\omega) \leq \alpha_n(\omega) < T_{nk}(\omega)$ for all $n \geq 1$ and $k \leq m_n$. Therefore, $S_{nk}(\omega) \leq D_A(\omega) < T_{nk}(\omega)$ for all $n \geq 1$ and all $k \leq m_n$ as well, proving that $D_A(\omega) \in A(\omega)$ and thus $(D_A(\omega), \omega) \in A$, as desired.

Next, we prove that D_A is a predictable stopping time. To this end, define $\mathcal{H}$ by putting $\mathcal{H} = \{S \in \mathcal{T}_p | S \leq D_A \text{ almost surely}\}$. We always have that $\mathcal{H}$ contains the constant zero, and so $\mathcal{H}$ is a nonempty family of variables. Therefore, by Theorem A.1.18, there exists an essential upper envelope T of $\mathcal{H}$, meaning a random variable T such that $S \leq T$ for all $S \in \mathcal{H}$ and if U is another random variable with this property, then $T \leq U$ almost surely. Also by Theorem A.1.18, there is a sequence of variables (S_n) in $\mathcal{H}$ such that $T = \sup_n S_n$ almost surely. By Lemma 2.1.9 and Lemma 2.1.8, T is a predictable stopping time. As the usual conditions hold, we may modify T on a set of measure zero to ensure that T is a predictable stopping time, that T is an essential upper envelope of $\mathcal{H}$ and that $T(\omega) \leq D_A(\omega)$ for all ω. We claim that T and D_A are equal almost surely, if we can prove this, Lemma 2.1.8 will allow us to conclude the proof.

Note that by construction, $T \leq D_A$. Thus, it will suffice to prove $D_A \leq T$ almost surely. To show this, we first show that A is equal to a set whose debut is more easily seen to be less than T. Defining $B_n = \cup_{k=1}^{m_n} [\![S_{nk} \vee T, T_{nk} \vee T[\![$, we may apply Lemma 2.1.5 to see that $B_n \in \mathbb{I}_p$. Furthermore, by considering the cases $T_{nk}(\omega) \leq T(\omega)$, $S_{nk}(\omega) \leq T(\omega) < T_{nk}(\omega)$ and $T(\omega) < S_{nk}(\omega)$ separately, we also obtain $B_n = \cup_{k=1}^{m_n} [\![S_{nk}, T_{nk}[\![\cap [\![T, \infty[\![= A_n \cap [\![T, \infty[\![$, proving that $\cap_{n=1}^{\infty} B_n = (\cap_{n=1}^{\infty} A_n) \cap [\![T, \infty[\![= A \cap [\![T, \infty[\![$. Now, as $T(\omega) \leq D_A(\omega)$ for all ω, we obtain $A \subseteq [\![T, \infty[\![$, and so $A \subseteq B_n$, proving $\cap_{n=1}^{\infty} B_n = A$. Furthermore, we find that the debut D_{B_n} is given by $D_{B_n} = \min\{(S_{nk} \vee T)_{(S_{nk} \vee T < T_{nk} \vee T)} | k \leq m_n\}$, which is a predictable stopping time by Lemma 2.1.5 and Lemma 2.1.10.

As $A \subseteq B_n$, it holds that $D_{B_n} \leq D_A$. As we also know that D_{B_n} is a predictable stopping time, we conclude that $D_{B_n} \in \mathcal{H}$, and therefore $D_{B_n} \leq T$ almost surely. On the other hand, as $S_{nk} \vee T \geq T$ for all $k \leq m_n$, we also find that $D_{B_n} \geq T$. Therefore, D_{B_n} is almost surely equal to T. Now fix $\omega \in \Omega$ such that $T(\omega)$ is finite and $D_{B_n}(\omega) = T(\omega)$ for all n, this is almost surely the case on $(T < \infty)$. As $(D_{B_n}(\omega), \omega) \in B_n$ for all n, we find that $(T(\omega), \omega) \in B_n$ for all n, and so $(T(\omega), \omega) \in A$. In particular, $D_A \leq T$. Thus, we conclude that $D_A \leq T$ almost surely. As we have by construction that $T \leq D_A$ almost surely as well, we conclude that $T = D_A$ almost surely. Therefore, Lemma 2.1.8 shows that D_A is a predictable stopping time, as desired. □

Theorem 2.1.12. *Let T be a stopping time. The following are equivalent:*

(1). T is predictable.

(2). $[\![T]\!]$ is a predictable set.

(3). $1_{[\![T,\infty[\![}$ is a predictable process.

Proof. First note that as $1_{[\![T,\infty[\![} = 1_{[\![T]\!]} + 1_{]\!]T,\infty[\![}$, where the process $1_{]\!]T,\infty[\![}$ is predictable for all stopping times because it is left-continuous and adapted, the equivalence between (2) and (3) follows immediately. And if T is predictable, Lemma 2.1.6 shows that $[\![T,\infty[\![$ is predictable, so $1_{[\![T,\infty[\![}$ is a predictable process. In order to complete the proof of the theorem, it therefore suffices to show that if $[\![T]\!]$ is a predictable set, then T is predictable.

To this end, assume that $[\![T]\!]$ is predictable. Let $A \in \Sigma^p$. Noting that T is $\mathcal{F}$-$\mathcal{B}_+$ measurable, we find that $\omega \mapsto (T(\omega),\omega)$ is $\mathcal{F}$-$\mathcal{B}_+ \otimes \mathcal{F}$ measurable. As $(t,\omega) \mapsto 1_A(t,\omega)$ is Σ^p-$\mathcal{B}_+$ measurable, it is in particular $\mathcal{B}_+ \otimes \mathcal{F}$-$\mathcal{B}_+$ measurable. Therefore, the composite mapping $\omega \mapsto 1_A(T(\omega),\omega)$ is $\mathcal{F}$-$\mathcal{B}_+$ measurable. From these observations, we conclude that by putting $\mu(A) = \int 1_A(T(\omega),\omega) 1_{(T<\infty)}(\omega)\,\mathrm{d}P(\omega)$ for any $A \in \Sigma^p$, we obtain a well-defined mapping from Σ^p to $[0,\infty]$. It is immediate that μ is a nonnegative measure concentrated on $[\![T]\!]$ and bounded by $P(T < \infty)$.

We now find that by Lemma A.1.8, there exists a sequence (A_n) in $(\mathbb{I}_p)_\delta$ with $A_n \subseteq [\![T]\!]$ such that $\mu([\![T]\!]) - \frac{1}{n} \le \mu(A_n) \le \mu([\![T]\!])$. Let S_n be the debut of A_n, we know from Lemma 2.1.11 that S_n is a predictable stopping time. Furthermore, we know that whenever $S_n(\omega)$ is finite, $(S_n(\omega),\omega) \in A_n$. Now define $T_n = \min\{S_1,\ldots,S_n\}$ and let $U = \inf_n T_n$. We claim that U is a predictable stopping time which is almost surely equal to T.

We first use Lemma 2.1.9 to show that U is a predictable stopping time. To this end, first note that T_n is decreasing. Also, note that if S_k is finite, we have $(S_k(\omega),\omega) \in A_k \subseteq [\![T]\!]$, so $S_k(\omega) = T(\omega)$. Therefore, $S_k(\omega)$ is equal to either $T(\omega)$ or ∞. As a consequence, T_n is also equal to either $T(\omega)$ or ∞. As T_n is decreasing, it follows that T_n is constant from a point onwards. As S_n is a predictable stopping time, so is T_n by Lemma 2.1.5, and Lemma 2.1.9 then shows that U is a predictable stopping time.

It remains to show that T and U are almost surely equal. To this end, note that

$$\begin{aligned} P(T \neq U) &= P(T \neq U, U = \infty) + P(T \neq U, U < \infty) \\ &= P(T \neq U, S_n = \infty \text{ for all } n) + P(T \neq U, S_n < \infty \text{ for some } n) \\ &= P(T < \infty, S_n = \infty \text{ for all } n) + P(T \neq U, S_n < \infty \text{ for some } n). \end{aligned}$$

We claim that each of the two terms above are zero. We first consider the second term. Recall that $T_n(\omega)$ is constant from a point onwards and is equal to either $T(\omega)$ or ∞. If $S_n(\omega) < \infty$ for some n, then $T_n(\omega)$ is finite and thus equal to $T(\omega)$, yielding $U(\omega) = T(\omega)$. This proves that $P(T \neq U, S_n < \infty \text{ for some } n) = 0$.

It remains to show $P(T < \infty, S_n = \infty \text{ for all } n) = 0$. To this end, fix some $k \ge 1$.

Note that when $S_k(\omega) = \infty$, it holds that $\{t \geq 0 \mid (t,\omega) \in A_k\}$ is empty. Now, if $T(\omega) < \infty$ and $(T(\omega),\omega) \in A_k$, $\{t \geq 0 \mid (t,\omega) \in A_k\}$ would not be empty. Thus, we conclude that either $T(\omega) = 0$ or $T(\omega) < \infty$ and $(T(\omega),\omega) \notin A_k$. Thus, as μ is concentrated on $[\![T]\!]$ and $A_k \subseteq [\![T]\!]$, we obtain

$$\begin{aligned} P(T < \infty, S_k = \infty) &= \int 1_{(T(\omega)<\infty, S_k=\infty)} \,\mathrm{d}P(\omega) \leq \int 1_{((T(\omega),\omega)\in A_k^c)} 1_{(T(\omega)<\infty)} \,\mathrm{d}P(\omega) \\ &= \mu(A_k^c) = \mu([\![T]\!] \cap A_k^c) = \mu([\![T]\!]) - \mu(A_k) \leq \tfrac{1}{k}. \end{aligned}$$

As it holds that $P(T < \infty, S_n = \infty \text{ for all } n) \leq P(T < \infty, S_k = \infty)$ for all $k \geq 1$, this implies $P(T < \infty, S_n = \infty \text{ for all } n) = 0$. We conclude that T and U are almost surely equal. Applying Lemma 2.1.8, we then find that T is predictable. □

An immediate useful application of Theorem 2.1.12 is the following.

Lemma 2.1.13. *Let T be a stopping time. Assume that there is a predictable set A such that T is the debut of A, and assume that whenever $T(\omega)$ is finite, then $(T(\omega),\omega) \in A$. Then T is predictable.*

Proof. Note that when $T(\omega) < \infty$, it holds that $(t,\omega) \notin A$ for $0 \leq t < T(\omega)$. Combining this with our assumption that $(T(\omega),\omega) \in A$ whenever $T(\omega) < \infty$, we obtain

$$\begin{aligned} [\![T]\!] &= \{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid t = T(\omega) < \infty\} \\ &= \{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid (t,\omega) \in A, t \leq T(\omega) < \infty\} \\ &= A \cap [\![0,T]\!]. \end{aligned}$$

Since $[\![0,T]\!] =]\!]T,\infty[\![^c$, the set $[\![0,T]\!]$ is predictable by is predictable by Lemma 2.1.3. And by our assumptions, A is predictable as well. We conclude that $[\![T]\!]$ is a predictable set, and so Theorem 2.1.12 shows that T is predictable. □

2.2 Stochastic processes and predictability

In this section, we investigate the connection between stochastic processes, the predictable σ-algebra and predictable stopping times. These results will pave the way for the results in Section 2.3, where we characterize predictable càdlàg processes in terms of their behaviour at particular stopping times. We begin by introducing a new σ-algebra related to a given stopping time.

Definition 2.2.1. *Let T be a stopping time. We define the strictly pre-T σ-algebra $\mathcal{F}_{T-}$ by putting $\mathcal{F}_{T-} = \sigma(\mathcal{F}_0 \cup \{F \cap (T > t) | t \geq 0, F \in \mathcal{F}_t\})$.*

We think of $\mathcal{F}_{T-}$ as the σ-algebra of events strictly prior to T.

Lemma 2.2.2. *Let S and T be stopping times.*

(1). T is $\mathcal{F}_{T-}$-measurable.

(2). $\mathcal{F}_{T-} \subseteq \mathcal{F}_T$.

(3). If $S \leq T$, then $\mathcal{F}_{S-} \subseteq \mathcal{F}_{T-}$.

(4). If $S \leq T$ and $S < T$ whenever $T > 0$, then $\mathcal{F}_S \subseteq \mathcal{F}_{T-}$.

(5). If T is predictable with announcing sequence (T_n), then $\mathcal{F}_{T-} = \sigma(\cup_{n=1}^{\infty} \mathcal{F}_{T_n})$.

Proof. **Proof of (1).** Since $(T > t) \in \mathcal{F}_{T-}$ for $t \geq 0$ by definition, it follows immediately that T is measurable with respect to $\mathcal{F}_{T-}$.

Proof of (2). Letting $F \in \mathcal{F}_0$, we have $F \cap (T \leq t) \in \mathcal{F}_t$ as $(T \leq t) \in \mathcal{F}_t$, so $\mathcal{F}_0 \subseteq \mathcal{F}_T$. Next, let $s \geq 0$ and $F \in \mathcal{F}_s$, we then obtain $F \cap (T > s) \cap (T \leq t) = \emptyset \in \mathcal{F}_t$ when $t \leq s$ and $F \cap (T > s) \cap (T \leq t) \in \mathcal{F}_t$ when $t > s$. Thus, $F \cap (T > s) \in \mathcal{F}_T$ and we have shown $\mathcal{F}_{T-} \subseteq \mathcal{F}_T$.

Proof of (3). Assume that we have $S \leq T$, let $t \geq 0$ and let $F \in \mathcal{F}_t$. We then obtain $F \cap (S > t) = F \cap (S > t) \cap (T > t)$, which is in $\mathcal{F}_{T-}$ since $F \cap (S > t) \in \mathcal{F}_t$. This shows $\mathcal{F}_{S-} \subseteq \mathcal{F}_{T-}$.

Proof of (4). Assume $S \leq T$ and $S < T$ when $T > 0$. Let $A \in \mathcal{F}_S$, so that $A \cap (S \leq t) \in \mathcal{F}_t$ for all $t \geq 0$. We find

$$\begin{aligned} A &= (A \cap (T = 0)) \cup (A \cap (T > 0)) \\ &= (A \cap (S = 0) \cap (T = 0)) \cup (A \cap (S < T)) \\ &= (A \cap (S = 0) \cap (T = 0)) \cup \cup_{q \in \mathbb{Q}_+} A \cap (S < q) \cap (q < T). \end{aligned}$$

Here, $A \cap (S = 0) \cap (T = 0) \in \mathcal{F}_0 \subseteq \mathcal{F}_{T-}$. Since $A \cap (S < q) \in \mathcal{F}_q$, $A \cap (S < q) \cap (q < T) \in \mathcal{F}_{T-}$. We conclude $A \in \mathcal{F}_{T-}$.

Proof of (5). Let T be predictable with announcing sequence (T_n). From the previous step, we know $\mathcal{F}_{T_n} \subseteq \mathcal{F}_{T-}$, so clearly $\sigma(\cup_{n=1}^\infty \mathcal{F}_{T_n}) \subseteq \mathcal{F}_{T-}$. On the other hand, letting $t \geq 0$ and $F \in \mathcal{F}_t$, we have $F \cap (T > t) = \cup_{n=1}^\infty F \cap (T_n > t) \in \sigma(\cup_{n=1}^\infty \mathcal{F}_{T_n-}) \subseteq \sigma(\cup_{n=1}^\infty \mathcal{F}_{T_n})$, so $\mathcal{F}_{T-} \subseteq \sigma(\cup_{n=1}^\infty \mathcal{F}_{T_n})$, as was to be proved. □

Lemma 2.2.3. *If T is a predictable stopping time and $F \in \mathcal{F}_{T-}$, then T_F is a predictable stopping time.*

Proof. Define $\mathcal{H} = \{F \in \mathcal{F}_{T-} | T_F \text{ is predictable }\}$. We will show that $\mathcal{H}$ is a σ-algebra containing a generator for $\mathcal{F}_{T-}$. In order to obtain that $\mathcal{H}$ is a σ-algebra, first note that clearly, $\Omega \in \mathcal{H}$. Assume $F \in \mathcal{H}$, then

$$\begin{aligned} [\![T_{F^c}]\!] &= [\![T]\!] \cap (\mathbb{R}_+ \times F^c) = [\![T]\!] \cap (\mathbb{R}_+ \times F)^c \\ &= [\![T]\!] \cap ([\![T]\!] \cap \mathbb{R}_+ \times F)^c = [\![T]\!] \cap [\![T_F]\!]^c. \end{aligned}$$

As both T and T_F are predictable by assumption, Theorem 2.1.12 shows that $[\![T]\!]$ and $[\![T_F]\!]^c$ are predictable. Therefore, $[\![T_{F^c}]\!]$ is predictable and thus T_{F^c} is a predictable stopping time, again by Theorem 2.1.12. Next, let (F_n) be a sequence in $\mathcal{H}$ and define $F = \cup_{n=1}^\infty F_n$. We then have $[\![T_F]\!] = [\![T]\!] \cap \mathbb{R}_+ \times F = [\![T]\!] \cap \cup_{n=1}^\infty \mathbb{R}_+ \times F_n = \cup_{n=1}^\infty [\![T_{F_n}]\!]$, and by the same arguments as before, this shows that $[\![T_F]\!]$ is a predictable stopping time. We conclude that $\mathcal{H}$ is a σ-algebra. Next, we show that $\mathcal{H}$ contains a generator for $\mathcal{F}_{T-}$. By definition, $\mathcal{F}_{T-}$ is generated by $\mathcal{F}_0$ and the sets $F \cap (T > s)$ where $s \geq 0$ and $F \in \mathcal{F}_s$. If $F \in \mathcal{F}_0$, $[\![T_F]\!] = [\![T]\!] \cap \mathbb{R}_+ \times F = [\![T]\!] \cap [\![0_F, \infty[\![$, which is predictable since 0_F is a predictable stopping time by Lemma 2.1.5, so $\mathcal{F}_0 \subseteq \mathcal{H}$. Let $s \geq 0$ and $F \in \mathcal{F}_s$, then

$$\begin{aligned} [\![T_{F \cap (T>s)}]\!] &= \{(t, \omega) \in \mathbb{R}_+ \times \Omega | t = T(\omega), \omega \in F \cap (T > s)\} \\ &= \{(t, \omega) \in \mathbb{R}_+ \times \Omega | t = T(\omega), t > s, \omega \in F\} = [\![T]\!] \cap]\!]s_F, \infty[\![. \end{aligned}$$

Since both $[\![T]\!]$ and $]\!]s_F, \infty[\![$ are predictable, $[\![T_{F \cap (T>s)}]\!]$ is predictable and so $T_{F \cap (T>s)}$ is predictable. Thus, $F \cap (T > s) \in \mathcal{H}$. We may now conclude $\mathcal{F}_{T-} \subseteq \mathcal{H}$, proving the lemma. □

The following useful lemma corresponds to a version of Lemma 1.1.11, using $\mathcal{F}_{T-}$ instead of $\mathcal{F}_T$.

Lemma 2.2.4. *Let S and T be stopping times.*

(1). If Z is $\mathcal{F}_S$ measurable, $Z 1_{(S<T)}$ is $\mathcal{F}_{T-}$ measurable.

(2). If Z is $\mathcal{F}_{S-}$ measurable and S is predictable, $Z1_{(S\leq T)}$ is $\mathcal{F}_{(S\wedge T)-}$ measurable.

(3). If Z is $\mathcal{F}_{S-}$ measurable and S and T are predictable, $Z1_{(S<T)}$ is $\mathcal{F}_{(S\wedge T)-}$ measurable.

In particular, if S and T are stopping times, $(S<T)$ is in $\mathcal{F}_{T-}$. If S is predictable, $(S\leq T)\in\mathcal{F}_{(S\wedge T)-}$, and $(S<T)$, $(S\leq T)$ and $(S=T)$ are in $\mathcal{F}_{T-}$. If S and T both are predictable, $(S<T)$, $(S\leq T)$ and $(S=T)$ are in $\mathcal{F}_{(S\wedge T)-}$.

Proof. **Proof of (1).** In this case, we merely assume that S and T are stopping times. Let Z be $\mathcal{F}_S$ measurable. Let B be a Borel set not containing zero. We then obtain that $(Z1_{(S<T)}\in B)=(Z\in B)\cap(S<T)=\cup_{q\in\mathbb{Q}_+}(Z\in B)\cap(S\leq q)\cap(q<T)$, which is in $\mathcal{F}_{T-}$, since $(Z\in B)\in\mathcal{F}_S$ such that $(Z\in B)\cap(S\leq q)\in\mathcal{F}_q$. This shows that $Z1_{(S<T)}$ is $\mathcal{F}_{T-}$ measurable.

Proof of (2). Now assume that S and T are stopping times, where S is predictable. Also assume that Z is $\mathcal{F}_{S-}$ measurable. We want to show that $Z1_{(S\leq T)}$ is $\mathcal{F}_{(S\wedge T)-}$ measurable. As $Z1_{(S\leq T)}=Z1_{(S\leq S\wedge T)}$, it suffices to show that $Z1_{(S\leq T)}$ is $\mathcal{F}_{T-}$ measurable. To this end, as in the proof of the first claim, it suffices to show that $F\cap(S\leq T)\in\mathcal{F}_{T-}$ for $F\in\mathcal{F}_{S-}$.

To prove this, define $\mathcal{H}=\{F\in\mathcal{F}_{S-}\mid F\cap(S\leq T)\in\mathcal{F}_{T-}\}$. It will suffice to argue that $\mathcal{H}$ is a Dynkin class containing a generator for $\mathcal{F}_{S-}$ which is stable under intersections. We begin by proving that $\mathcal{H}$ contains such a generator. As S is predictable, it holds by Lemma 2.2.2 that $\cup_{n=1}^\infty\mathcal{F}_{S_n}$ is a generator for $\mathcal{F}_{S-}$, where (S_n) denotes an announcing sequence for S. We will prove that $\cup_{n=1}^\infty\mathcal{F}_{S_n}$ is in $\mathcal{H}$.

To this end, fix $m\geq 1$ and let $F\in\mathcal{F}_{S_m}$. We will prove that $F\cap(S\leq T)\in\mathcal{F}_{T-}$. Note that $F\cap(S\leq T)=(F\cap(S\leq T)\cap(S>0))\cup(F\cap(S=0))$. Here, $F\cap(S=0)\in\mathcal{F}_0\subseteq\mathcal{F}_{T-}$, so it suffices to show $F\cap(S\leq T)\cap(S>0)\in\mathcal{F}_{T-}$. As S_n is strictly less than S on $(S>0)$, we find $F\cap(S\leq T)\cap(S>0)=\cap_{k=m}^\infty F\cap(S>0)\cap(S_k<T)$. As (S_n) is increasing, $F\in\mathcal{F}_{S_k}$ whenever $k\geq m$. As $(S>0)\in\mathcal{F}_0$, $F\cap(S>0)\in\mathcal{F}_{S_k}$ as well, and so, by what was already proven, $F\cap(S>0)\cap(S_k<T)\in\mathcal{F}_{T-}$. From this, we conclude $F\cap(S\leq T)\cap(S>0)\in\mathcal{F}_{T-}$. From this, we obtain $\mathcal{F}_{S_m}\subseteq\mathcal{H}$. As a consequence, $\cup_{n=1}^\infty\mathcal{F}_{S_n}\subseteq\mathcal{H}$.

It remains to show that $\mathcal{H}$ is a Dynkin class. As $\cup_{n=1}^\infty\mathcal{F}_{S_n}\subseteq\mathcal{H}$, we find $\Omega\in\mathcal{H}$. It is immediate that $\mathcal{H}$ is stable under countable unions. Next, assume that $F,G\in\mathcal{H}$ with $F\subseteq G$, we want to argue that $G\setminus F\in\mathcal{H}$. We have

$$(G\setminus F)\cap(S\leq T)=G\cap(S\leq T)\cap(F\cap(S\leq T))^c.$$

Here, $G \cap (S \leq T)$ and $F \cap (S \leq T)$ are in $\mathcal{F}_{T-}$ by asumption, so $(G \setminus F) \cap (S \leq T)$ is in $\mathcal{F}_{T-}$ as well, yielding $G \setminus F \in \mathcal{H}$. We conclude that $\mathcal{H}$ is a Dynkin class, so Lemma A.1.1 now allows us to conclude that $F \cap (S \leq T) \in \mathcal{F}_{T-}$ for all $F \in \mathcal{F}_{S-}$, as desired.

Proof of (3). Finally, we consider the case where both S and T are predictable. We are to show that $Z1_{(S<T)}$ is $\mathcal{F}_{(S\wedge T)-}$ measurable. As in the previous cases, it suffices to show that for $F \in \mathcal{F}_{S-}$, it holds that $F \cap (S < T)$ is in $\mathcal{F}_{(S\wedge T)-}$. To this end, first note that $F \cap (S < T) = F \cap (S \leq T) \cap (S < T)$. From our previous results, we find that as S is predictable, $F \cap (S \leq T) \in \mathcal{F}_{(S\wedge T)-}$, and as T is predictable, $(S < T) = (S \geq T)^c \in \mathcal{F}_{(S\wedge T)-}$. The result follows.

Proof of remaining claims. From (1), it follows that $(S < T) \in \mathcal{F}_{T-}$ when S and T are stopping times. Now assume in additaion that S is predictable. From (2) it follows that $(S \leq T) \in \mathcal{F}_{(S\wedge T)-}$. And from Lemma 2.2.2, it follows that $(S \leq T) \in \mathcal{F}_{T-}$. Therefore, $(S = T) \in \mathcal{F}_{T-}$ as well. Finally, if S and T both are predictable, it follows from (2) and (3) that $(S < T)$ and $(S \leq T)$ are in $\mathcal{F}_{(S\wedge T)-}$, and so $(S = T)$ is in $\mathcal{F}_{(S\wedge T)-}$ as well. □

Using the previous results, we may now obtain some fundamental results on the interplay between predictable processes and the strictly pre-T σ-algebra $\mathcal{F}_{T-}$.

Lemma 2.2.5. *If T is a stopping time and Z is $\mathcal{F}_T$ measurable, then $Z1_{]\!]T,\infty[\![}$ is predictable. If T is a predictable stopping time and Z is $\mathcal{F}_{T-}$ measurable, then $Z1_{[\![T,\infty[\![}$ is predictable.*

Proof. Assume first that T is a stopping time. For any $F \in \mathcal{F}_T$, T_F is a stopping time as well, so since $1_F 1_{]\!]T,\infty]\!]} = 1_{]\!]T_F,\infty[\![}$, we find that $1_F 1_{]\!]T,\infty[\![}$ is predictable as $]\!]T_F,\infty[\![$ is predictable by Theorem 2.1.3. By stability of measurability under elementary operations and pointwise limits, it follows that $Z1_{]\!]T,\infty[\![}$ is predictable whenever Z is $\mathcal{F}_T$ measurable.

Now assume that T is a predictable stopping time. For any $F \in \mathcal{F}_{T-}$, we obtain by Lemma 2.2.3 that T_F is a predictable stopping time. As $1_F 1_{[\![T,\infty[\![} = 1_{[\![T_F,\infty[\![}$ and the latter is predictable by Theorem 2.1.12, we conclude that for any $F \in \mathcal{F}_{T-}$, $1_F 1_{[\![T,\infty[\![}$ is predictable. Again by stability of measurability under elementary operations and pointwise limits, we obtain that $Z1_{]\!]T,\infty[\![}$ is predictable whenever Z is $\mathcal{F}_{T-}$ measurable. □

Lemma 2.2.6. *Let T be a stopping time. If X is a predictable process, then $X_T 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$-measurable.*

Proof. First consider the case where $X = 1_{[\![S,\infty[\![}$ for some predictable stopping time S. In this case, $X_T 1_{(T<\infty)} = 1_{(S \le T<\infty)}$. By Lemma 2.2.4, $(S \le T)$ is $\mathcal{F}_{T-}$ measurable, and by Lemma 2.2.2, $(T < \infty)$ is $\mathcal{F}_{T-}$ measurable. Therefore, we conclude that $X_T 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$ measurable in this case.

Now let $\mathcal{H}$ be the class of $B \in \Sigma^p$ such that with $X = 1_B$, $X_T 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$ measurable for all stopping times T. We have just shown that $\mathcal{H}$ contains $[\![S, \infty[\![$ for all stopping times S, and by Lemma 2.1.6, this family is a generating class for Σ^p, stable under intersections. Therefore, if we can show that $\mathcal{H}$ is a Dynkin class, it follows that $\mathcal{H} = \Sigma^p$. It is immediate that $\mathbb{R}_+ \times \Omega \in \mathcal{H}$. That $\mathcal{H}$ is stable under set subtraction and increasing unions then follow from the stability properties of measurability under elementary operations. We conclude that $\mathcal{H}$ is a Dynkin class and so by Lemma A.1.1, it holds that whenever $X = 1_B$ and $B \in \Sigma^p$, $X_T 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$ measurable. By approximation with simple Σ^p measurable functions, we now obtain the result of the lemma. □

Lemma 2.2.7. *Let X be a càdlàg adapted process, and let T be predictable. Then $X_{T-} 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$ measurable.*

Proof. Defining $Y = X_-$, it holds that $X_{T-} 1_{(T<\infty)} = Y_T 1_{(T<\infty)}$. As Y is càglàd and adapted, it is predictable. Therefore, the result follows from Lemma 2.2.6. □

Lemma 2.2.8. *Let X be a predictable process, and let T be a stopping time. Then the stopped process X^T is predictable as well.*

Proof. Note that $X^T = X 1_{[\![0,T]\!]} + X_T 1_{(T<\infty)} 1_{]\!]T,\infty[\![}$. The first term is predictable, since X is predictable and $[\![0,T]\!] =]\!]T,\infty[\![^c$ is predictable according to Lemma 2.1.3. The second term is predictable since $X_T 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$-measurable by Lemma 2.2.6, and so $X_T 1_{(T<\infty)} 1_{]\!]T,\infty[\![}$ is predictable by Lemma 2.2.5. □

2.3 Accessible and totally inaccessible stopping times

Using the results of Section 2.2, we now introduce accessible and totally inaccessible stopping times and use these to characterize càdlàg predictable processes in terms of their behaviour at jump times.

Definition 2.3.1. *We say that a stopping time T is totally inaccessible if it holds for any predictable stopping time S that $P(T = S < \infty) = 0$. We say that a stopping time T is* *accessible if there exists a sequence of predictable stopping times* (T_n) *with the property that* $[\![T]\!] \subseteq \cup_{n=1}^{\infty}[\![T_n]\!]$.

The following result shows that any stopping time can be decomposed into an accessible and a totally inaccessible part. Recall that for two sets $F, G \subseteq \Omega$, we define the symmetric difference $F\Delta G$ by putting $F\Delta G = (F \setminus G) \cup (G \setminus F)$.

Theorem 2.3.2. *Let* T *be any stopping time. There exists a set* $F \in \mathcal{F}_{T-}$ *such that we have* $T = T_F \wedge T_{F^c}$, *where* T_F *is accessible and* T_{F^c} *is totally inaccessible. The set* F *is almost surely unique in the sense that if* G *is another such set, then* $P((T < \infty) \cap F\Delta G) = 0$.

Proof. We define $\mathcal{H} = \{\cup_{n=1}^{\infty}(T = S_n < \infty)|(S_n) \text{ is a sequence in } \mathcal{T}_p\}$. By Lemma 2.2.4, we know that if T is a stopping time and S is a predictable stopping time, then $(T = S) \in \mathcal{F}_{T-}$. As we have $(T < \infty) \in \mathcal{F}_{T-}$ by Lemma 2.2.2, we conclude that $\cup_{n=1}^{\infty}(T = S_n < \infty) \in \mathcal{F}_{T-}$ for any sequence for any sequence $(S_n) \subseteq \mathcal{T}_p$, and so $\mathcal{H} \subseteq \mathcal{F}_{T-}$. Define $\alpha = \sup\{P(F)|F \in \mathcal{H}\}$. It then holds that for each n, there is $F_n \in \mathcal{H}$ such that $P(F_n) \geq \alpha - \frac{1}{n}$. Put $F = \cup_{n=1}^{\infty}F_n$, then $F \in \mathcal{H}$ and $F \in \mathcal{F}_{T-}$. As $F \in \mathcal{H}$, we have in particular that $F = \cup_{n=1}^{\infty}(T = S_n < \infty)$ for some sequence (S_n) in $\mathcal{T}_p$. Also, as $P(F) \geq P(F_n) \geq \alpha - \frac{1}{n}$ for all n, we conclude that $P(F) = \sup\{P(G) \mid G \in \mathcal{H}\}$. We claim that T_F is accessible and T_{F^c} is totally inaccessible.

To prove that T_F is accessible, we simply note that

$$
\begin{aligned}
[\![T_F]\!] &= \{(t,\omega) \in \mathbb{R}_+ \times \Omega | t = T(\omega), \omega \in F\} \\
&= \{(t,\omega) \in \mathbb{R}_+ \times \Omega | t = T(\omega),\ \exists\, n \in \mathbb{N} : T(\omega) = S_n(\omega) < \infty\} \subseteq \cup_{n=1}^{\infty}[\![S_n]\!],
\end{aligned}
$$

so T_F is accessible. To prove that T_{F^c} is totally inaccessible, let S be some predictable stopping time. Note that $F \cup (T = S < \infty) \in \mathcal{H}$ and we have

$$
\begin{aligned}
P(F \cup (T = S < \infty)) &= P(F) + P(F^c \cap (T = S < \infty)) \\
&= P(F) + P(T_{F^c} = S < \infty).
\end{aligned}
$$

As we have $P(F) = \sup\{P(G) \mid G \in \mathcal{H}\}$ by construction and $F \cup (T = S < \infty) \in \mathcal{H}$, we obtain $P(F \cup (T = S < \infty)) \leq P(F)$. And as $F \subseteq F \cup (T = S < \infty)$, also have $P(F) \leq P(F \cup (T = S < \infty))$. Thus, $P(F \cap (T = S < \infty)) = P(F)$, and the above then yields $P(T_{F^c} = S < \infty) = 0$, proving that T_{F^c} is totally inaccessible. This shows existence of the decomposition.

It remains to prove uniqueness. Assume that we have two decompositions into accessible and totally inaccessible parts, $T = T_F \wedge T_{F^c}$ and $T = T_G \wedge T_{G^c}$. We wish to show that

$P((T<\infty)\cap(F\Delta G))=0$. By symmetry it will suffice to prove, say, $P((T<\infty)\cap F\cap G^c)=0$. Assume that $P((T<\infty)\cap F\cap G^c)>0$. With $[\![T_F]\!]\subseteq\cup_{n=1}^\infty[\![S_n]\!]$ where S_n is predictable, we then obtain

$$\begin{aligned} 0 &< P((T<\infty)\cap F\cap G^c) \\ &\le P((\,\exists\, n: T=S_n<\infty)\cap F\cap G^c) \\ &\le P(\,\exists\, n: T_{G^c}=S_n<\infty), \end{aligned}$$

so there is some n such that $P(T_{G^c}=S_n<\infty)>0$, a contradiction with our assumption. We conclude $P((T<\infty)\cap F\cap G^c)=0$. □

Next, we work towards a result allowing us to decompose a sequence of stopping times into disjoint parts. The following extension of Lemma 2.1.8 will be useful in this regard.

Lemma 2.3.3. *Let S and T be two nonnegative variables and assume that S and T are equal almost surely. If T is a stopping time, so is S. If T is predictable, so is S. If T is accessible, so is S. If T is totally inaccessible, so is S.*

Proof. By Lemma 2.1.8, S is a stopping time if T is a stopping time, and S is predictable if T is predictable. Assume that T is accessible, and let (T_n) be a sequence of predictable stopping times such that $[\![T]\!]\subseteq\cup_{n=1}^\infty[\![T_n]\!]$. Define $U=S_{(T\neq S)}$. Since T and S are equal almost surely, $S_{(T\neq S)}$ is almost surely equal to infinity, therefore predictable by Lemma 2.1.8, and $[\![S]\!]\subseteq[\![T]\!]\cup[\![U]\!]\subseteq[\![U]\!]\cup\cup_{n=1}^\infty[\![T_n]\!]$, showing that S is accessible. Assume finally that T is totally inaccessible, and let R be any predictable stopping time. As T and S are almost surely equal, $P(S=R<\infty)=P(T=R<\infty)=0$, so S is totally inaccessible. □

Lemma 2.3.4. *Let (T_n) be a sequence of stopping times. There exists a sequence of stopping times (S_n) with disjoint graphs such that $\cup_{n=1}^\infty[\![T_n]\!]=\cup_{n=1}^\infty[\![S_n]\!]$. If the (T_n) are predictable, the (S_n) can be taken to be predictable as well.*

Proof. Assuming that (T_n) is any sequence of stopping times, we define $S_1=T_1$ and recursively $F_n=\cap_{k=1}^{n-1}(T_k\neq T_n)$ and $S_n=(T_n)_{F_n}$. Then $F_n\in\mathcal{F}_{T_n}$ by Lemma 1.1.11, so S_n is a stopping time. If $k<n$ with $S_k(\omega)<\infty$ and $S_n(\omega)<\infty$, we have $S_k(\omega)=T_k(\omega)$ and $\omega\in\cap_{i=1}^{n-1}(T_k\neq T_n)$, so $S_k(\omega)\neq S_n(\omega)$. Thus, the graphs $[\![S_k]\!]$ and $[\![S_n]\!]$ are disjoint. It remains to prove $\cup_{n=1}^\infty[\![T_n]\!]=\cup_{n=1}^\infty[\![S_n]\!]$. It is immediate that $[\![S_n]\!]\subseteq[\![T_n]\!]$, so the inclusion towards the left holds. Assume conversely that $(t,\omega)\in\cup_{n=1}^\infty[\![T_n]\!]$. Then, there exists a smallest $n\geq 1$ with $(t,\omega)=(T_n(\omega),\omega)$. In this case, $T_n(\omega)\neq T_k(\omega)$ for $k<n$, so $\omega\in F_n$

and thus $T_n(\omega) = S_n(\omega)$, yielding $(t,\omega) \in [\![S_n]\!]$. This proves the inequality towards the right, and so $\cup_{n=1}^\infty [\![T_n]\!] = \cup_{n=1}^\infty [\![S_n]\!]$ holds.

Turning to the predictable case, assume that each T_n is predictable. By Lemma 2.2.4, $(T_k \neq T_n) \in \mathcal{F}_{(T_k \wedge T_n)-} \subseteq \mathcal{F}_{T_n-}$, so $F_n \in \mathcal{F}_{T_n-}$ and by Lemma 2.2.3, S_n is predictable. □

Definition 2.3.5. *A regular sequence of stopping times is a sequence (T_n) of stopping times such that the graphs of the stopping times are disjoint and each T_n is either predictable or totally inaccessible.*

Lemma 2.3.6. *Let (T_n) be any sequence of stopping times. There exists a regular sequence of stopping times (R_n) such that $\cup_{n=1}^\infty [\![T_n]\!] \subseteq \cup_{n=1}^\infty [\![R_n]\!]$.*

Proof. By Lemma 2.3.4, we may assume that the (T_n) have disjoint graphs. By Theorem 2.3.2, there exists $F_n \in \mathcal{F}_{T_n-}$ such that with $S_n = (T_n)_{F_n}$ and $U_n = (T_n)_{F_n}$, S_n is accessible and U_n is totally inaccessible, and $T_n = S_n \wedge U_n$. We then have $[\![U_n]\!] \subseteq [\![T_n]\!]$, so the (U_n) have disjoint graphs as well. Next, as S_n is accessible, there exists for each $n \geq 1$ a sequence $(S_n^k)_{k\geq 1}$ of predictable stopping times such that $[\![S_n]\!] \subseteq \cup_{k=1}^\infty [\![S_n^k]\!]$. Lemma 2.3.4 then yields a sequence of predictable stopping times (V_n) with disjoint graphs such that $\cup_{n=1}^\infty \cup_{k=1}^\infty [\![S_n^k]\!] = \cup_{n=1}^\infty [\![V_n]\!]$. We then have $\cup_{n=1}^\infty [\![T_n]\!] \subseteq (\cup_{n=1}^\infty [\![V_n]\!]) \cup (\cup_{n=1}^\infty [\![U_n]\!])$, where each of the stopping times are either predictable or totally inaccessible, and it holds that (V_n) has disjoint graphs and (U_n) has disjoint graphs.

Next, note that $P(S_n = U_k < \infty) = 0$ for any n and k, since S_n is predictable and U_k is totally inaccessible. In particular, the set $F_n = \cup_{k=1}^\infty (S_n = U_k)$ is a null set. Therefore, by Lemma 2.1.8, $(S_n)_{F_n^c}$ is predictable, and we find $[\![(S_n)_{F_n^c}]\!] = [\![S_n]\!] \setminus \cup_{k=1}^\infty [\![U_k]\!]$. Therefore, we obtain

$$\begin{aligned}
\cup_{n=1}^\infty [\![T_n]\!] &\subseteq (\cup_{n=1}^\infty [\![S_n]\!]) \cup (\cup_{n=1}^\infty [\![U_n]\!]) \\
&= (\cup_{n=1}^\infty ([\![S_n]\!] \setminus \cup_{k=1}^\infty [\![U_k]\!])) \cup (\cup_{n=1}^\infty [\![U_n]\!]) \\
&= \left(\cup_{n=1}^\infty [\![(S_n)_{F_n^c}]\!]\right) \cup (\cup_{n=1}^\infty [\![U_n]\!]).
\end{aligned}$$

By construction, all the stopping times are disjoint, and each is either predictable or totally inaccessible. This proves the result. □

Lemma 2.3.6 allows us to prove Theorem 2.3.8, which states that the jumps of càdlàg adapted processes can be covered by a countable sequence of positive stopping times which are either predictable or totally inaccessible and which never take the same values, and if the process is predictable, all the stopping times can be taken to be predictable. This result will in several important cases allow us to restrict our analysis of jump time behaviour to jumps occurring at predictable or totally inaccessible jumps. To prove the theorem, we need the following lemma.

Lemma 2.3.7. *Let X be a process which is càdlàg and predictable. Let U be an open set in $\mathbb{R}$ such that $U \cap [-\varepsilon, \varepsilon] = \emptyset$ for some $\varepsilon > 0$. Define $T = \inf\{t \geq 0 \mid \Delta X_t \in U\}$. Then T is a predictable stopping time.*

Proof. From Lemma 1.1.14, we know that T is a stopping time. We need to prove that it is predictable. With $A = \{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid \Delta X_t(\omega) \in U\}$, it holds that T is the debut of A. Note that as X by Lemma A.2.3 pathwisely only has finitely many jumps greater than ε on bounded intervals, our assumption on U ensures that $\Delta X_T \in U$ whenever T is finite. Also, A is predictable as ΔX is predictable. Lemma 2.1.13 then shows that T is predictable. □

Theorem 2.3.8. *Let X be a càdlàg adapted process. There is a regular sequence of positive stopping times (T_n) such that $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} \subseteq \cup_{n=1}^\infty [\![T_n]\!]$. If X is predictable, then each T_n can be taken to be predictable.*

Proof. By Lemma 1.1.15, defining $T_1^k = \inf\{t \geq 0 \mid |\Delta X_t| > \frac{1}{k}\}$ for $k \geq 1$, and recursively for $n \geq 2$, $T_n^k = \inf\{t > T_{n-1}^k \mid |\Delta X_t| > \frac{1}{k}\}$, T_n^k is a stopping time for all $k \geq 1$ and $n \geq 1$, $|\Delta X_{T_n^k}| > \frac{1}{k}$ whenever T_n^k is finite and $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} = \cup_{k=1}^\infty \cup_{n=1}^\infty [\![T_n^k]\!]$. Then, applying Lemma 2.3.6, we obtain the existence of a regular sequence of stopping times (R_n) such that $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} \subseteq \cup_{n=1}^\infty [\![R_n]\!]$. Putting $T_n = (R_n)_{(R_n > 0)}$, we find by Lemma 2.1.5 that (T_n) is a regular sequence of positive stopping times, and as no process jumps at time zero, we get $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} \subseteq \cup_{n=1}^\infty [\![T_n]\!]$, as desired.

Now consider the case where X is predictable. Note that with $Y^{kn} = \sum_{i=1}^{n-1} \Delta X_{T_i^k} 1_{[\![T_i^k, \infty[\![}$, we also have $T_n^k = \inf\{t \geq 0 \mid |\Delta(X - Y^{kn})_t| > \frac{1}{k}\}$, where $X - Y^{kn}$ is càdlàg and predictable by Lemma 2.2.6 and Lemma 2.2.5. Therefore, by Lemma 2.3.7, T_k^n is predictable. Applying Lemma 2.3.4, we obtain a sequence of predictable stopping times (S_n) with disjoint graphs such that $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} = \cup_{n=1}^\infty [\![S_n]\!]$. Putting $T_n = (S_n)_{(S_n > 0)}$, Lemma 2.1.5 allows us to conclude that (T_n) is a sequence of positive predictable stopping times, and as in the previous case, $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} \subseteq \cup_{n=1}^\infty [\![T_n]\!]$ since no process jumps at time zero. □

Note that in the following theorem, we make use of our convention that $\Delta X_\infty = 0$ for all càdlàg processes, yielding $\Delta X_T = \Delta X_T 1_{(T<\infty)}$ for all stopping times T. This convention allows us to formulate our result without the use of unwieldy indicator functions.

Theorem 2.3.9. *Let X be an adapted càdlàg process. Then X is predictable if and only if it holds that for every totally inaccessible stopping time T, $\Delta X_T = 0$ almost surely, and for every predictable stopping time T, ΔX_T is $\mathcal{F}_{T-}$ measurable.*

Proof. First assume that X is predictable. By Theorem 2.3.8, there exists a sequence of predictable times (T_n) such that $\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} \subseteq \cup_{n=1}^\infty [\![T_n]\!]$. Let T be a totally inaccessible stopping time. As T_n is predictable, we obtain $P(T = T_n < \infty) = 0$ for $n \geq 1$, and so we find $P(\Delta X_T \neq 0) \leq \sum_{n=1}^\infty P(T = T_n < \infty) = 0$. Thus, for every totally inaccessible stopping time T, $\Delta X_T = 0$ almost surely. Next, consider a predictable stopping time T. By Lemma 2.2.6, $X_T 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$ measurable, and by Lemma 2.2.7, $X_{T-} 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$ measurable. Therefore, $\Delta X_T 1_{(T<\infty)}$ and thus ΔX_T is $\mathcal{F}_{T-}$ measurable. We conclude that for every predictable stopping time T, ΔX_T is $\mathcal{F}_{T-}$ measurable. This proves the implication towards the right of the theorem.

Conversely, assume that X satisfies the two requirements in the statement of the theorem. By Theorem 2.3.8, there exists sequences (S_n) and (T_n) of stopping times with disjoint graphs such that $\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid |\Delta X_t| \neq 0\} \subseteq (\cup_{n=1}^\infty [\![S_n]\!]) \cup (\cup_{n=1}^\infty [\![T_n]\!])$, where S_n is predictable and T_n is totally inaccessible. By assumption, ΔX_{T_n} is almost surely zero. Put $U_n = (T_n)_{(\Delta X_{T_n} \neq 0)}$, it then holds that $\Delta X_{T_n} 1_{[\![T_n]\!]} = \Delta X_{U_n} 1_{[\![U_n]\!]}$, and U_n is predictable since it is almost surely infinite. We obtain

$$\begin{aligned} X &= X_- + \Delta X = X_- + \sum_{n=1}^\infty \Delta X_{S_n} 1_{[\![S_n]\!]} + \sum_{n=1}^\infty \Delta X_{T_n} 1_{[\![T_n]\!]} \\ &= X_- + \sum_{n=1}^\infty \Delta X_{S_n} 1_{[\![S_n]\!]} + \sum_{n=1}^\infty \Delta X_{U_n} 1_{[\![U_n]\!]}. \end{aligned}$$

Here, X_- is predictable because it is càglàd and adapted, and the second and third terms are predictable by Lemma 2.2.5. Consequently, X is predictable. $\square$

Theorem 2.3.9 is the main result of this section. Its usefulness is that it will allow us to check predictability of an adapted càdlàg process merely by analyzing its behaviour at stopping times. Two simple yet important consequence are given as the following.

Lemma 2.3.10. *Let X and Y be adapted càdlàg processes. Assume that X is predictable and that Y is a modification of X. Then Y is predictable as well.*

Proof. For any stopping time T, it holds that $\Delta X_T = \Delta Y_T$ almost surely. Therefore, Y satisfies the criteria of Theorem 2.3.9, and is therefore predictable. $\square$

Lemma 2.3.11. *Let (X^n) be a sequence of predictable càdlàg processes. Let Y be an adapted càdlàg process, and assume that for all stopping times T, ΔX^n_T converges almost surely to ΔY_T. Then Y is predictable as well.*

Proof. Fix any predictable stopping time T. By our assumptions, ΔX^n_T converges almost surely to ΔY_T. As X^n is predictable, Theorem 2.3.9 shows that ΔX^n_T is $\mathcal{F}_{T-}$ measurable. Therefore, ΔY_T is $\mathcal{F}_{T-}$ measurable as well. Next, fix any totally inaccessible stopping time T. By Theorem 2.3.9, ΔX^n_T is almost surely zero. Therefore, ΔY_T is almost surely zero as well. We conclude that Y satisfies the criteria of Theorem 2.3.9. Therefore, Y is predictable. $\square$

2.4 Exercises

Exercise 2.4.1. *Let Σ^o denote the optional σ-algebra, meaning the σ-algebra on $\mathbb{R}_+ \times \Omega$ generated by the adapted càdlàg processes. Show that $\Sigma^p \subseteq \Sigma^o \subseteq \Sigma^\pi$.*

Exercise 2.4.2. *Let X be a continuous adapted process with initial value zero, let $a \in \mathbb{R}$ with $a \neq 0$ and let $T = \inf\{t \geq 0 \mid X_t = a\}$. Argue that T is a predictable stopping time, and find an announcing sequence for T.*

Exercise 2.4.3. *Let S and T be two predictable stopping times. Show that the equality $\mathcal{F}_{(S\wedge T)-} = \mathcal{F}_{S-} \wedge \mathcal{F}_{T-}$ holds.*

Exercise 2.4.4. *Let T be some variable. Assume that there exists a sequence (T_n) of stopping times such that $T_n < T$ whenever $T > 0$ for all $n \geq 1$ and such that $\sup_n T_n = T$. Show that T is a predictable stopping time.*

Exercise 2.4.5. *Assume that S is a stopping time. Let T be a stopping time such that $T \geq S$ and $T > S$ whenever S is finite. Assume that $\mathcal{F}_T = \mathcal{F}_S$. Show that T is a predictable stopping time.*

Exercise 2.4.6. *Show that $\mathcal{F}_{T-} = \sigma(\{X_T \mid X \text{ is predictable }\})$ whenever T is a finite predictable stopping time.*

Exercise 2.4.7. *Show that $\mathcal{F}_{T-} = \sigma(\{X_{T-} \mid X \text{ is càdlàg adapted }\})$ for any finite stopping time T.*

Exercise 2.4.8. *Let T be a stopping time taking only countably many values. Show that T is accessible.*

Exercise 2.4.9. *Let T be an accessible stopping time. Show that T is predictable if and only if $(T = S) \in \mathcal{F}_{S-}$ for all predictable stopping times S.*

Exercise 2.4.10. *Let M be a càdlàg adapted process with initial value zero and assume that M_t is almost surely convergent. Show that $M \in \mathcal{M}^u$ if and only if M_T is integrable with $EM_T = 0$ for all accessible stopping times T.*

Exercise 2.4.11. *Let T be a totally inaccessible stopping time. Show that there exists a sequence (T_n) of accessible stopping times such that T_n converges to T from above.*

Chapter 3

Local martingales

In this chapter, we introduce local martingales, which essentially are processes which are martingales when stopped at appropriate stopping times. Local martingales function as a natural generalization of martingales which later will be seen to behave particularly pleasantly as integrators. During the course of this chapter, the results of the previous two chapters will be applied together to gain a coherent understanding of the space of local martingales. This understanding will allow us in Chapter 4 to define the stochastic integral of a predictable process with respect to a local martingale in a simple and elegant manner.

The chapter is structured as follows. In Section 3.1, we formally introduce local martingales, and we prove some basic stability properties. Already at this point, we will be able to use the results on predictability from Chapter 2 to prove nontrivial results.

In Section 3.2, we consider the problem of characterizing the structure of local martingales with paths of finite variation. This understanding will be important in Section 3.3, where we show that any local martingale can be decomposed into a locally bounded component and a component of finite variation. Combining this result with our previous results from Chapter 1, we are able to construct the quadratic variation and quadratic covariation processes, which are fundamental tools for working with local martingales.

Finally, in Section 3.4, we use the quadratic covariation process to introduce the space of purely discontinuous local martingales, which is a sort of orthogonal complement to the space of local martingales with continuous paths. We prove that any local martingales can be decomposed uniquely as the sum of a continuous and purely discontinuous local martingale. This will be useful for our later construction of the stochastic integral with respect to a local martingale.

3.1 The space of local martingales

In this section, we introduce the basic results on local martingales, a convenient extension of the concept of martingales. We say that an increasing sequence of stopping times tending almost surely to infinity is a localising sequence. We then say that a process M is a local martingale if M is adapted and there is a localising sequence (T_n) such that M^{T_n} is a martingale for all n, and in this case, we say that (T_n) is a localising sequence for M. The space of càdlàg local martingales with initial value zero is denoted by $\mathcal{M}_\ell$. The space of continuous elements of $\mathcal{M}_\ell$ is denoted by $\mathbf{c}\mathcal{M}_\ell$.

Lemma 3.1.1. *It holds that $\mathcal{M}^b \subseteq \mathcal{M}^2 \subseteq \mathcal{M}^u \subseteq \mathcal{M} \subseteq \mathcal{M}_\ell$.*

Proof. It is immediate that $\mathcal{M}^b \subseteq \mathcal{M}^2$. By Lemma A.3.4, $\mathcal{M}^2 \subseteq \mathcal{M}^u$, and by construction we have $\mathcal{M}^u \subseteq \mathcal{M}$. If $M \in \mathcal{M}$, $M^T \in \mathcal{M}$ for any stopping time by Lemma 1.2.7, and so $\mathcal{M} \subseteq \mathcal{M}_\ell$. □

Lemma 3.1.2. *Let (S_n) and (T_n) be localising sequences. Then $(S_n \wedge T_n)$ is a localising sequence as well. If $M, N \in \mathcal{M}_\ell$, with localising sequences (S_n) and (T_n), then $(S_n \wedge T_n)$ is a localising sequence for both M and N.*

Proof. As $S_n \wedge T_n$ is a stopping time by Lemma 1.1.9 and $S_n \wedge T_n$ clearly tends almost surely to infinity, $(S_n \wedge T_n)$ is a localising sequence. Now assume that $M, N \in \mathcal{M}_\ell$ with localising sequences (T_n) and (S_n), respectively. Then $M^{T_n \wedge S_n} = (M^{T_n})^{S_n}$ is a martingale by Lemma 1.2.7, and so $(T_n \wedge S_n)$ is a localising sequence for M. Analogously, $(T_n \wedge S_n)$ is also a localising sequence for N. □

Lemma 3.1.3. *$\mathcal{M}_\ell$ is a vector space. If T is any stopping time and $M \in \mathcal{M}_\ell$, then $M^T \in \mathcal{M}_\ell$ as well. If $F \in \mathcal{F}_0$ and $M \in \mathcal{M}_\ell$, then $1_F M$ is in $\mathcal{M}_\ell$ as well, where $1_F M$ denotes the process $(1_F M)_t = 1_F M_t$.*

Proof. Let $M, N \in \mathcal{M}_\ell$ and let $\alpha, \beta \in \mathbb{R}$. Using Lemma 3.1.2, let (T_n) be a localising sequence for both M and N. Then $(\alpha M + \beta N)^{T_n} = \alpha M^{T_n} + \beta N^{T_n}$ is a martingale, so

$\alpha M + \beta N \in \mathcal{M}_\ell$ and $\mathcal{M}_\ell$ is a vector space. As regards the stopped process, let $M \in \mathcal{M}_\ell$ and let T be any stopping time. Let (T_n) be a localising sequence for M. As $M^{T_n} \in \mathcal{M}$, we obtain that $(M^T)^{T_n} = (M^{T_n})^T \in \mathcal{M}$, proving that (T_n) is also a localising sequence for M^T, so that $M^T \in \mathcal{M}_\ell$. Finally, let $M \in \mathcal{M}_\ell$ and $F \in \mathcal{F}_0$. Let (T_n) be a localising sequence such that $M^{T_n} \in \mathcal{M}$. For any bounded stopping time T, $E1_F M_T^{T_n} = E1_F(EM_T^{T_n}|\mathcal{F}_0) = 0$ by Theorem 1.2.6, so by Lemma 1.2.8, $1_F M^{T_n}$ is a martingale. As $(1_F M)^{T_n} = 1_F M^{T_n}$, $1_F M$ is in $\mathcal{M}_\ell$. □

Lemma 3.1.4. *Let M and N be two càdlàg adapted processes with initial value zero. If M and N are indistinguishable and $M \in \mathcal{M}_\ell$, then $N \in \mathcal{M}_\ell$ as well.*

Proof. Let (T_n) be a localising sequence for M. Then M^{T_n} is a martingale. As N^{T_n} is indistinguishable from M^{T_n}, we obtain that N^{T_n} is a martingale as well. As N is càdlàg and has initial value zero, we conclude that $N \in \mathcal{M}_\ell$. □

The following lemma shows that each local martingale is not only locally a martingale, but is locally a uniformly integrable martingale. Also, Lemma 3.1.6 shows that a continuous local martingale also is locally a continuous bounded martingale, and Lemma 3.1.7 shows that a process which locally is a local martingale in fact is a local martingale. Lemma 3.1.7 includes a result for the case where the localisation is of the form $M^{T_n}1_{(T_n>0)}$ instead of M^{T_n}, this will be useful in the course of Chapter 4.

Lemma 3.1.5. *Let $M \in \mathcal{M}_\ell$. Then there exists a localising sequence (T_n) such that for each n, $M^{T_n} \in \mathcal{M}^u$.*

Proof. Let T_n be a sequence such that M^{T_n} is a martingale for $n \geq 1$. Then $M^{T_n \wedge n}$ is in $\mathcal{M}^u$ by Theorem 1.2.4, since it is a martingale convergent almost surely and in $\mathcal{L}^1$ to $M_{T_n \wedge n}$. This proves the result. □

Lemma 3.1.6. *Let $M \in \mathbf{c}\mathcal{M}_\ell$. There exists a localising sequence (T_n) such that M^{T_n} is in $\mathbf{c}\mathcal{M}^b$ for all n. In particular, $M^{T_n} \in \mathbf{c}\mathcal{M}^2$ and $M^{T_n} \in \mathbf{c}\mathcal{M}^u$.*

Proof. Let $T_n = \inf\{t \geq 0 || M_t| > n\}$. By Lemma 1.1.17, (T_n) is a localising sequence, and M^{T_n} is bounded. And by Lemma 1.2.7, M^{T_n} is a continuous martingale. Thus, M^{T_n} is a bounded continuous martingale. Clearly, M^{T_n} is then bounded in $\mathcal{L}^2$, so we also obtain $M^{T_n} \in \mathbf{c}\mathcal{M}^2$ and $M^{T_n} \in \mathbf{c}\mathcal{M}^u$. □

Lemma 3.1.7. *Let M be a càdlàg adapted process with initial value zero. If there is a localising sequence (T_n) such that M^{T_n} is in $\mathcal{M}_\ell$ for all $n \geq 1$, then $M \in \mathcal{M}_\ell$. If there is a localising sequence (T_n) such that $M^{T_n} 1_{(T_n > 0)}$ is in $\mathcal{M}_\ell$ for all $n \geq 1$, then $M \in \mathcal{M}_\ell$.*

Proof. First consider the case where M^{T_n} is in $\mathcal{M}_\ell$ for all $n \geq 1$. Let (T^n_k) be a localising sequence such that $(M^{T_n})^{T^n_k}$ is in $\mathcal{M}$. Fix $n \geq 1$, then T^n_k tends to infinity almost surely as k tends to infinity. Therefore, it also holds that $\lim_k P(|T^n_k| \leq M) = 0$ for all $M > 0$. For each $n \geq 1$, choose k_n such that $P(|T^n_{k_n}| \leq n) \leq 1/2^n$. Then $\sum_{n=1}^\infty P(|T^n_{k_n}| \leq n)$ is finite, so the Borel-Cantelli lemma yields that $\cap_{n=1}^\infty \cup_{i=n}^\infty (|T^i_{k_i}| \leq i)$ has probability zero. Therefore, $T^n_{k_n}$ converges almost surely to infinity. Now put $S_n = \max_{1 \leq i \leq n}\{T_i \wedge T^i_{k_i}\}$. Then S_n is a localising sequence, and $M^{S_n} \in \mathcal{M}$ for each $n \geq 1$. Thus, $M \in \mathcal{M}_\ell$.

Next, assume that $M^{T_n} 1_{(T_n > 0)}$ is in $\mathcal{M}_\ell$ for all $n \geq 1$. Define $S_n = (T_n)_{(T_n > 0)} \wedge 0_{(T_n = 0)}$. As $(T_n > 0) \in \mathcal{F}_0$, (S_n) is a sequence of stopping times. As (T_n) almost surely increases to infinity, the set family $(T_n > 0)_{n \geq 1}$ is increasing and the set family $(T_n = 0)_{n \geq 1}$ is decreasing, and it almost surely holds that $T_n > 0$ eventually. Therefore, (S_n) also increases almost surely to infinity. Thus, (S_n) is a localising sequence. And as M has initial value zero, we obtain

$$M^{S_n}_t = M^{T_n}_t 1_{(T_n > 0)} + M^0_t 1_{(T_n = 0)} = M^{T_n}_t 1_{(T_n > 0)}.$$

Therefore, the results already proven yields that $M \in \mathcal{M}_\ell$, as desired. □

Recall that we in Lemma 1.2.11 proved that a martingale with paths of finite variation which is also continuous in fact is evanescent. We will now use our understanding of predictability to prove a considerable extension of this result, namely that any martingale with paths of finite variation which is also predictable in fact is continuous and therefore evanescent.

Lemma 3.1.8. *Let $M \in \mathcal{M}^u$ and let T be a predictable stopping time. Then ΔM_T is integrable, and $E(\Delta M_T | \mathcal{F}_{T-}) = 0$.*

Proof. We first show that ΔM_T is integrable. Let T_n be an announcing sequence for T. We then find $M_{T-} = \lim_n M_{T_n}$, where the convergence is almost sure. As $M_{T_n} = E(M_\infty | \mathcal{F}_{T_n})$ by Theorem 1.2.4, $(M_{T_n})_{n \geq 1}$ is uniformly integrable, and therefore we have convergence in $\mathcal{L}^1$ as well. As a consequence, we obtain in particular that M_{T-} is integrable, and as M_T is integrable by the optional sampling theorem, we conclude that ΔM_T is integrable. In order to obtain the second result of the lemma, recall from Lemma 2.2.2 that $\mathcal{F}_{T-} = \sigma(\cup_{n=1}^\infty \mathcal{F}_{T_n})$. As $E(M_T | \mathcal{F}_{T_n}) = M_{T_n}$, we find that M_{T_n} converges almost surely and in $\mathcal{L}^1$ to $E(M_T | \mathcal{F}_{T-})$.

As we also have convergence to M_{T-}, we conclude $E(M_T|\mathcal{F}_{T-}) = M_{T-}$ by uniqueness of limits. As M_{T-} is $\mathcal{F}_{T-}$ measurable, this shows $E(\Delta M_T|\mathcal{F}_{T-}) = 0$, as desired. $\square$

Theorem 3.1.9. *Let $M \in \mathcal{M}_\ell$. If M is predictable, M is almost surely continuous. If M is predictable or almost surely continuous with paths of finite variation, M is evanescent.*

Proof. We first show that if $M \in \mathcal{M}_\ell$ is predictable, then it is almost surely continuous. To do so, first assume that $M \in \mathcal{M}^u$ and that M is predictable. Let T be any predictable stopping time. Applying Theorem 2.3.9 and Lemma 3.1.8, ΔM_T is integrable and it holds that $\Delta M_T = E(\Delta M_T|\mathcal{F}_{T-}) = 0$ almost surely. By Theorem 2.3.8, there exists a sequence of predicable stopping times (T_n) covering the jumps of M. By what we already have shown, ΔM_{T_n} is almost surely zero for each $n \geq 1$. Therefore, M is almost surely continuous. Next, consider the case where $M \in \mathcal{M}_\ell$. By Lemma 3.1.5, there is a localising sequence (T_n) such that $M^{T_n} \in \mathcal{M}^u$. By Lemma 2.2.8, M^{T_n} is predictable as well. Therefore, M^{T_n} is almost surely continuous. Letting n tend to infinity, we conclude that M is almost surely continuous. This shows that any $M \in \mathcal{M}_\ell$ which is predictable is almost surely continuous.

It remains to prove that if M is predictable or almost surely continuous with paths of finite variation, M is evanescent. We first show that if $M \in \mathcal{M}_\ell$ has paths of finite variation and is continuous, then M is evanescent. Consider such an M. Using Lemma 3.1.6, let (T_n) be a localising sequence for M such that $M^{T_n} \in \mathbf{c}\mathcal{M}$. Then M^{T_n} also has paths of finite variation, so by Lemma 1.2.11, M^{T_n} is evanescent. As T_n tends to infinity, we conclude that M is evanescent as well. In the case where M only is almost surely continuous, let F be the null set where M is not continuous. Putting $N = 1_{F^c}M$, $N \in \mathcal{M}_\ell$ by Lemma 3.1.4, N has paths of finite variation and N is continuous. Therefore, by what was already shown, N is evanescent. As M is a modification of N, M is evanescent as well. Finally, assume that M is predictable with paths of finite variation. From what we already have shown, M is almost surely continuous, and so M is evanescent. This concludes the proof. $\square$

3.2 Finite variation processes and compensators

In Chapter 1, we introduced the following spaces: $\mathcal{V}$ is the space of adapted càdlàg processes with initial value zero and paths of finite variation, $\mathcal{A}$ is the subspace of increasing elements of $\mathcal{V}$, $\mathcal{A}^i$ is the subspace of integrable elements of $\mathcal{A}$ and $\mathcal{V}^i$ is the subspace of integrable elements of $\mathcal{V}$, meaning elements such that the variation process is integrable. We now introduce two further spaces of this type. Let $A \in \mathcal{V}$. We say that A is locally integrable if there exists a localizing sequence (T_n) such that $A^{T_n} \in \mathcal{V}^i$ for each $n \geq 1$. We denote the space of locally integrable elements of $\mathcal{V}$ by $\mathcal{V}^i_\ell$. We denote the subspace of increasing elements of $\mathcal{V}^i_\ell$ by $\mathcal{A}^i_\ell$. It then also holds that $\mathcal{A}^i_\ell$ is the space of elements of $\mathcal{A}$ such that there exists a localizing sequence (T_n) with the property that $A^{T_n} \in \mathcal{A}^i$ for each $n \geq 1$.

In this section, we will show that for any process $A \in \mathcal{V}^i_\ell$, there exists a predictable process $\Pi^*_p A \in \mathcal{V}^i_\ell$, unique up to evanescence, such that $A - \Pi^*_p A$ is in $\mathcal{M}_\ell$. We refer to the mapping $\Pi^*_p : \mathcal{V}^i_\ell \to \mathcal{V}^i_\ell$ defined up to evanescence as the compensating projection, and we refer to $\Pi^*_p A$ as the compensator of A. The compensator will allow us to give a characterization of elements of $\mathcal{M}_\ell$ with paths of finite variation.

The proof of the existence of the compensator is somewhat technical. We begin by establishing some lemmas. First we prove the existence of the compensator for a particularly simple type of elements of $\mathcal{V}^i_\ell$, namely processes of the form $\xi 1_{[\![T,\infty[\![}$, where T is a positive stopping time and ξ is nonnegative, bounded and $\mathcal{F}_T$ measurable. Afterwards, we apply monotone convergence arguments and localisation arguments to obtain the general existence result.

Lemma 3.2.1. *Let T be a positive stopping time and let ξ be nonnegative, bounded and $\mathcal{F}_T$ measurable. Define $A = \xi 1_{[\![T,\infty[\![}$. The process A is then an element of $\mathcal{A}^i$, and there exists a predictable process $\Pi^*_p A$ in $\mathcal{A}^i$ such that $A - \Pi^*_p A$ is a uniformly integrable martingale.*

Proof. It is immediate that $A \in \mathcal{A}^i$. To prove the existence of the compensator, our strategy will be to consider discrete-time compensators for finer and finer dyadic partitions of $\mathbb{R}_+$. Let $t^n_k = k2^{-n}$ for $k, n \geq 0$. We define

$$\begin{aligned} A^n_t &= A_{t^n_k} \text{ for } t^n_k \leq t < t^n_{k+1} \text{ and} \\ B^n_t &= \sum_{i=1}^{k+1} E(A_{t^n_i} - A_{t^n_{i-1}} | \mathcal{F}_{t^n_{i-1}}) \text{ for } t^n_k < t \leq t^n_{k+1}, \end{aligned}$$

and $B^n_0 = 0$. Note that since T is positive, both A^n and B^n have initial value zero. Also note that A^n is càdlàg adapted and B^n is càglàd adapted. Put $M^n = A^n - B^n$. Note that M^n is adapted, but not necessarily càdlàg or càglàd. Also note that, with the convention that a sum over an empty index set is zero, it holds that

$$A^n_{t^n_k} = A_{t^n_k} \quad \text{and} \quad B^n_{t^n_k} = \sum_{i=1}^{k} E(A_{t^n_i} - A_{t^n_{i-1}} | \mathcal{F}_{t^n_{i-1}}).$$

Therefore, $(B_{t^n_k})_{k\geq 0}$ is the compensator of the discrete-time increasing process $(A_{t^n_k})_{k\geq 0}$, so $(M^n_{t^n_k})_{k\geq 0}$ is a discrete-time martingale with initial value zero. Let $c > 0$ be a bound for ξ. By Lemma A.4.7, $B^n_{t^n_k}$ is square-integrable and it holds that $E(B^n_{t^n_k})^2 \leq 2c^2$. Thus, $E(M^n_{t^n_k})^2 \leq 4E(A^n_{t^n_k})^2 + 4E(B^n_{t^n_k})^2 \leq 12c^2$. We conclude that $(M^n_{t^n_k})_{k\geq 0}$ is bounded in $\mathcal{L}^2$, and so by Lemma A.4.4 convergent almost surely and in $\mathcal{L}^2$ to a square-integrable limit M^n_∞, and the sequence $(M^n_\infty)_{n\geq 0}$ is bounded in $\mathcal{L}^2$ as well.

By Lemma A.3.7, there exists a sequence of naturals (K_n) with $K_n \geq n$ and for each n a finite sequence of reals $\lambda^n_n, \ldots, \lambda^n_{K_n}$ in the unit interval summing to one such that $\sum_{i=n}^{K_n} \lambda^n_i M^i_\infty$ is convergent in $\mathcal{L}^2$ to some variable M_∞. By Theorem 1.3.3, it then holds that there is $M \in \mathcal{M}^2$ such that $E\sup_{t\geq 0}(M_t - \sum_{i=n}^{K_n} \lambda^n_i M^i_t)^2$ tends to zero, M is then a càdlàg version of the process $t \mapsto E(M_\infty|\mathcal{F}_t)$. By picking a subsequence and relabeling, we may assume that in addition to the properties already noted, $\sup_{t\geq 0}(M_t - \sum_{i=n}^{K_n} \lambda^n_i M^i_t)^2$ also converges almost surely to zero.

Define $B = A - M$. Note that as A and M both are càdlàg and adapted, B is càdlàg and adapted, and it is immediate that $A - B$ is in $\mathcal{M}^u$. Therefore, if we can show that B has a modification which is increasing and predictable, the proof of existence will be concluded.

We are now in a position to outline the remainder of the proof. Put $C^n = \sum_{i=n}^{K_n} \lambda^n_i B^i$. Note that C^n is càglàd, adapted and increasing. In particular, C^n is predictable. Define $\mathbb{D}_+ = \{k2^{-n} \mid k \geq 0, n \geq 0\}$. The remainder of the proof will proceed in three parts: First, we show that $B_q = \lim_{n\to\infty} C^n_q$ almost surely for all $q \in \mathbb{D}_+$, this will allow us to conclude that B is almost surely increasing. Secondly, we prove that $B_t = \limsup_{n\to\infty} C^n_t$ almost surely, simultaneuously for all $t \geq 0$, this will allow us to show that B has a predictable modification. Thirdly, we collect our conclusions to obtain existence of the compensator.

Step 1. ***B* is almost surely increasing.** Note that for each $q \in \mathbb{D}_+$, it holds that $A_q = \lim_{n\to\infty} A^n_q$ pointwise. Therefore,

$$B_q = A_q - M_q = \lim_{n\to\infty} A^n_q - \sum_{i=n}^{K_n} \lambda^n_i M^i_q = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda^n_i B^i_q = \lim_{n\to\infty} C^n_q,$$

almost surely. From this, we obtain that B is almost surely increasing on $\mathbb{D}_+$. Recalling that $B = A - M$ so that B is càdlàg, this allows us to conclude that B is almost surely increasing on all of $\mathbb{R}_+$.

Step 2. $B_t = \limsup_{n\to\infty} C^n_t$ **simultaneously.** Next, we show that almost surely, it holds that $B_t = \limsup_{n\to\infty} C^n_t$ for all $t \geq 0$, this will allow us to show that B has a predictable modification. To this end, note that for $t \geq 0$ and $q \in \mathbb{D}_+$ with $t \leq q$, $\limsup_{n\to\infty} C^n_t \leq \limsup_{n\to\infty} C^n_q = B_q$. As B is càdlàg, this yields $\limsup_{n\to\infty} C^n_t \leq B_t$.

This holds almost surely for all $t \in \mathbb{R}_+$ simultaneously. Similarly, $\liminf_{n\to\infty} C^n_t \geq B_{t-}$ almost surely, simultaneously for all $t \geq 0$. All in all, we conclude that almost surely, $B_t = \limsup_{n\to\infty} C^n_t$ for all continuity points t of B, simultaneously for all $t \geq 0$. As the jumps of B can be exhausted by a countable sequence of stopping times, we find that in order to show the desired result on the limes superior, it suffices to show for any stopping time S that $B_S = \limsup_{n\to\infty} C^n_S$. To do so, first note that

$$\lim_{t\to\infty} C^n_t = \lim_{m\to\infty} C^n_m = \lim_{m\to\infty} \sum_{i=n}^{K_n} \lambda^n_i B^i_m = \lim_{m\to\infty} A_m - \sum_{i=n}^{K_n} \lambda^n_i M^i_m = A_\infty - \sum_{i=n}^{K_n} \lambda^n_i M^i_\infty,$$

so C^n has an almost sure limit C^n_∞, which is integrable, and by our earlier bounds, we obtain

$$\begin{aligned} \|C^n_\infty\|_2 &\leq \|A_\infty\|_2 + \left\| \sum_{i=n}^{K_n} \lambda^n_i M^i_\infty \right\|_2 \leq \|A_\infty\|_2 + \sum_{i=n}^{K_n} \lambda^n_i \left\| M^i_\infty \right\|_2 \\ &= (EA^2_\infty)^{1/2} + \sum_{i=n}^{K_n} \lambda^n_i (E(M^i_\infty)^2)^{1/2} \leq (1+\sqrt{12})c, \end{aligned}$$

so $(C^n_\infty)_{n\geq 1}$ is bounded in $\mathcal{L}^2$. Now fix a stopping time S. We first note that as we have $0 \leq C^n_S \leq C^n_\infty$, the sequence of variables $(C^n_S)_{n\geq 0}$ is bounded in $\mathcal{L}^2$ and thus in particular uniformly integrable. Therefore, $\limsup_{n\to\infty} EC^n_S \leq E \limsup_{n\to\infty} C^n_S \leq EB_S$ by Lemma A.3.8. As $\limsup_{n\to\infty} C^n_S \leq B_S$ almost surely, we conclude that in order to show that $\limsup_{n\to\infty} C^n_S = B_S$ almost surely, it suffices to show that EC^n_S converges to EB_S.

To this end, define a stopping time S_n by putting $S_n = \infty$ whenever $S = \infty$ and putting $S_n = t^n_k$ whenever $t^n_{k-1} < S \leq t^n_k$. Then (S_n) is a sequence of stopping times taking values in $\mathbb{D}_+$ and infinity and converging downwards to S. Note that for all $n \geq 1$, it holds that $B^n_S = \sum_{k=0}^\infty B^n_{t^n_{k+1}} 1_{(t^n_k < S \leq t^n_{k+1})} = \sum_{k=0}^\infty B^n_{t^n_{k+1}} 1_{(S_n = t^n_{k+1})} = B^n_{S_n}$. Also, $A^n_{S_n} = A_{S_n}$. Therefore, recalling that $(A^n_{t^n_k} - B^n_{t^n_k})_{k\geq 0}$ is a uniformly integrable martingale for all $n \geq 1$, we obtain

$$EC^n_S = E\sum_{i=n}^{K_n} \lambda^n_i B^i_S = \sum_{i=n}^{K_n} \lambda^n_i EB^i_{S_i} = \sum_{i=n}^{K_n} \lambda^n_i EA^i_{S_i} = \sum_{i=n}^{K_n} \lambda^n_i EA_{S_i}.$$

As A is càdlàg and bounded, and S_n converges downwards to S, the dominated convergence theorem allows us to obtain that EA_{S_n} converges to EA_S. Therefore, $\sum_{i=n}^{K_n} \lambda^n_i EA_{S_i}$ also converges to EA_S. Combining this with the above and recalling that $A - B \in \mathcal{M}^u$, we conclude that $\lim_{n\to\infty} EC^n_S = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda^n_i EA_{S_i} = EA_S = EB_S$. Recalling our earlier observations, we may now conclude that $\limsup_{n\to\infty} C^n_t = B_t$ almost surely for all points of discontinuity of B, and so all in all, the result holds almost surely for all $t \in \mathbb{R}_+$ simultaneously.

Step 3. The existence proof. We now collect our conclusions to obtain the existence of the compensator. Using the two previous steps, let F be the almost sure set where B is increasing and $B = \limsup_{n\to\infty} C^n$. Put $\Pi_p^* A = B 1_F$. We claim that $\Pi_p^* A$ satisfies the requirements to be the compensator of A.

To prove this, first note that by Lemma 2.3.10, $1_F C^n$ is a predictable càdlàg process. As $\Pi_p^* A = \limsup_{n\to\infty} 1_F C^n$, $\Pi_p^* A$ is predictable. Also, it is immediate that $\Pi_p^* A$ is increasing. And as $A - \Pi_p^* A$ is a modification of $A - B$, $A - \Pi_p^* A$ is a uniformly integrable martingale. We conclude that $\Pi_p^* A$ satisfies all the requirements to be the compensator of A. □

With Lemma 3.2.1 in hand, the remainder of the proof for the existence of the compensator merely consists of monotone convergence arguments.

Lemma 3.2.2. *Let A^n be a sequence of processes in $\mathcal{A}^i$ such that $\sum_{n=1}^\infty A^n$ converges pointwise to a process A. Assume for each $n \geq 1$ that B^n is a predictable element of $\mathcal{A}^i$ such that $A^n - B^n$ is a uniformly integrable martingale. The process A is then in $\mathcal{A}^i$, and $\sum_{n=1}^\infty B^n$ almost surely converges pointwise to a predictable process $\Pi_p^* A$ in $\mathcal{A}^i$ such that $A - \Pi_p^* A$ is a uniformly integrable martingale.*

Proof. Clearly, A is in $\mathcal{A}^i$. With $B = \sum_{n=1}^\infty B^n$, B is a well-defined process with values in $[0, \infty]$, since each B^n is nonnegative. We wish to argue that there is a modification of B which is the compensator of A. First note that as each B^n is increasing and nonnegative, so is B. Also, as $A^n - B^n$ is a uniformly integrable martingale, the optional sampling theorem and two applications of the monotone convergence theorem yields for any bounded stopping time T that

$$EB_T = \lim_{n\to\infty} \sum_{k=1}^n EB_T^k = \lim_{n\to\infty} \sum_{k=1}^n EA_T^k = EA_T,$$

which in particular shows that B almost surely takes finite values. Therefore, by Lemma A.2.7, we obtain that B is almost surely nonnegative, càdlàg and increasing. Also, by another two applications of the monotone convergence theorem, we obtain for any stopping time T that $EB_T = \lim_{t\to\infty} EB_{T\wedge t} = \lim_{t\to\infty} EA_{T\wedge t} = EA_T$. This holds in particular with $T = \infty$, and therefore, the limit of B_t as t tends to infinity is almost surely finite and is furthermore integrable. Lemma A.2.7 then also shows that $\sum_{k=1}^n B^k$ converges almost surely uniformly to B on $\mathbb{R}_+$.

We now let $\Pi_p^* A$ be a nonnegative càdlàg increasing adapted modification of B. Then $\Pi_p^* A$ is in $\mathcal{A}^i$, and $E(\Pi_p^* A)_T = EA_T$ for all stopping times T, so by Lemma 1.2.8, $A - \Pi_p^* A$ is a uniformly integrable martingale. Also, $\sum_{k=1}^n B^k$ almost surely converges uniformly to

$\Pi_p^* A$ on $\mathbb{R}_+$. Therefore, Lemma 2.3.11 shows that $\Pi_p^* A$ is predictable. This concludes the proof. $\square$

Theorem 3.2.3. *Let $A \in \mathcal{V}_\ell^i$. There exists a predictable process $\Pi_p^* A$ in $\mathcal{V}_\ell^i$, unique up to indistinguishability, such that $A - \Pi_p^* A$ is a local martingale. If $A \in \mathcal{A}^i$, $\Pi_p^* A$ is in $\mathcal{A}^i$, if $A \in \mathcal{V}^i$, $\Pi_p^* A$ is in $\mathcal{V}^i$ and if $A \in \mathcal{A}_\ell^i$, $\Pi_p^* A$ is in $\mathcal{A}_\ell^i$. Also, $A - \Pi_p^* A$ is a uniformly integrable martingale when $A \in \mathcal{V}^i$.*

Proof. We first consider uniqueness. If $A \in \mathcal{V}_\ell^i$ and B and C are two predictable processes in $\mathcal{V}_\ell^i$ such that $A - B$ and $A - C$ both are local martingales, we find that $B - C$ is a predictable local martingale with paths of finite variation. By Theorem 3.1.9, uniqueness follows.

As for existence, we first consider the case where $A = \xi 1_{[\![T,\infty[\![}$ with T a positive stopping time, $\xi \in \mathcal{L}^1(\mathcal{F}_T)$ and $\xi \geq 0$. There exists a sequence of simple, nonnegative and $\mathcal{F}_T$ measurable variables ξ_n converging upwards to ξ. We can assume without loss of generality that $\xi_1 = 0$. Define A^n by putting $A^n = (\xi_{n+1} - \xi_n) 1_{[\![T,\infty[\![}$, then $A^n \in \mathcal{A}^i$, and $\sum_{k=1}^n A^k$ converges pointwise to A. Furthermore, by Lemma 3.2.1, there exists a predictable process B^n in $\mathcal{A}^i$ such that $A^n - B^n$ is a uniformly integrable martingale. By Lemma 3.2.2, we then find that there also exists a predictable process $\Pi_p^* A$ in $\mathcal{A}^i$ such that $A - \Pi_p^* A$ is a uniformly integrable martingale.

Now consider a general element $A \in \mathcal{A}^i$. By Theorem 2.3.8, there exists a sequence of positive stopping times (T_n) with disjoint graphs covering the jumps of A. For $n \geq 1$, define $A^n = \Delta A_{T_n} 1_{[\![T_n,\infty[\![}$, A^n is then an element of $\mathcal{A}^i$. Also define $A^d = \sum_{n=1}^\infty \Delta A_{T_n} 1_{[\![T_n,\infty[\![}$. As $A \in \mathcal{A}^i$, A^d is a well-defined element of $\mathcal{A}^i$. Furthermore, $A - A^d$ is a continuous element of $\mathcal{A}^i$. By the results already shown, there exists predictable processes B^n in $\mathcal{A}^i$ with the property that $\Delta A_{T_n} 1_{[\![T_n,\infty[\![} - B^n$ is a uniformly integrable martingale. As $\sum_{k=1}^n A^k$ converges pointwise to A^d, we find by Lemma 3.2.2 that there exists a predictable process $\Pi_p^* A^d$ in $\mathcal{A}^i$ such that $A^d - \Pi_p^* A^d$ is a uniformly integrable martingale. Putting $\Pi_p^* A = A - A^d + \Pi_p^* A^d$, we find that since $A - A^d$ is a predictable element of $\mathcal{A}^i$, $\Pi_p^* A$ is a predictable element of $\mathcal{A}^i$, and $A - \Pi_p^* A$ is a uniformly integrable martingale, proving existence for the case where $A \in \mathcal{A}^i$.

Next, assume $A \in \mathcal{V}^i$. Using Lemma 1.4.1, we may decompose the process A as $A = A^+ - A^-$, where $A^+, A^- \in \mathcal{A}^i$. Putting $\Pi_p^* A = \Pi_p^* A^+ - \Pi_p^* A^-$, we obtain that $\Pi_p^* A$ is a predictable element of $\mathcal{V}^i$ such that $A - \Pi_p^* A$ is a uniformly integrable martingale. Finally, we consider the case where $A \in \mathcal{V}_\ell^i$. In this case, there is a localising sequence (T_n) such that A^{T_n} is in $\mathcal{V}^i$. From what was already shown, there is a process $\Pi_p^* A^{T_n}$, unique up to indistinguishability, such that $A^{T_n} - \Pi_p^* A^{T_n}$ is a uniformly integrable martingale. In particular, it holds that $A^{T_n} - (\Pi_p^* A^{T_{n+1}})^{T_n} = (A^{T_{n+1}} - \Pi_p^* A^{T_{n+1}})^{T_n}$ is a uniformly integrable martingale, and so $(\Pi_p^* A^{T_{n+1}})^{T_n} = \Pi_p^* A^{T_n}$ up to indistinguishability. Therefore, the processes $\Pi_p^* A^{T_n}$ may be pasted together to a process $\Pi_p^* A$ in $\mathcal{V}_\ell^i$ such that for all $n \geq 1$, $(\Pi_p^* A)^{T_n} = \Pi_p^* A^{T_n}$ almost surely. In particular, it almost surely holds that $\Delta \Pi_p^* A_t = \lim_{n\to\infty} \Delta \Pi_p^* A_t^{T_n}$ for all $t \geq 0$. Lemma 2.3.11 then shows that $\Pi_p^* A$ is predictable. As $(A - \Pi_p^* A)^{T_n}$ is a uniformly integrable martingale, $A - \Pi_p^* A$ is a local martingale. This completes the proof of existence.

As regards the properties of $\Pi_p^* A$, we have already shown that when $A \in \mathcal{A}^i$, we have $\Pi_p^* A \in \mathcal{A}^i$, and when $A \in \mathcal{V}^i$, we have $\Pi_p^* A \in \mathcal{V}^i$. If $A \in \mathcal{A}_\ell^i$, we may take a localising sequence such that $A^{T_n} \in \mathcal{A}^i$ and obtain $\Pi_p^* A^{T_n} \in \mathcal{A}^i$. By uniqueness, we have $(\Pi_p^* A)^{T_n} = \Pi_p^* A^{T_n}$ up to indistinguishability, so that $\Pi_p^* A \in \mathcal{A}_\ell^i$. $\square$

Theorem 3.2.3 establishes existence and uniqueness of the compensating projection mapping. Next, we prove some basic properties of the compensator.

Lemma 3.2.4. *Let $A, B \in \mathcal{V}_\ell^i$. Then, the following holds up to evanescence.*

1. *Π_p^* maps $\mathcal{V}^i$ into $\mathcal{V}^i$, $\mathcal{A}^i$ into $\mathcal{A}^i$ and $\mathcal{A}_\ell^i$ into $\mathcal{A}_\ell^i$. If $A \in \mathcal{V}^i$, $A - \Pi_p^* A \in \mathcal{M}^u$.*
2. *For $\alpha, \beta \in \mathbb{R}$, $\Pi_p^*(\alpha A + \beta B) = \alpha \Pi_p^* A + \beta \Pi_p^*$.*
3. *$\Pi_p^*(\Pi_p^* A) = \Pi_p^* A$.*
4. *For any stopping time T, $(\Pi_p^* A)^T = \Pi_p^* A^T$.*

Proof. The first property is part of Theorem 3.2.3. Let $\alpha, \beta \in \mathbb{R}$ and let $A, B \in \mathcal{V}_\ell^i$. We then find that $\alpha A + \beta B - (\Pi_p^* A + \beta \Pi_p^* B)$ is in $\mathcal{M}_\ell$, so by uniqueness, $\Pi_p^*(\alpha A + \beta B) = \alpha \Pi_p^* A + \beta \Pi_p^*$ up to evanescence. Also, for $A \in \mathcal{V}_\ell^i$, as $\Pi_p^* A$ is predictable, we have that $\Pi_p^* A$ satisfies the requirements for being the compensator of $\Pi_p^* A$. Finally, let T be some stopping time and let $A \in \mathcal{V}_\ell^i$. Then $A^T - (\Pi_p^* A)^T = (A - \Pi_p^* A)^T \in \mathcal{M}_\ell$. By Lemma 2.2.8, $(\Pi_p^* A)^T$ is predictable, so we obtain $(\Pi_p^* A)^T = \Pi_p^* A^T$ up to evanescence, as desired. $\square$

Lemma 3.2.5. *Let $A \in \mathcal{V}_\ell^i$. If A only jumps at totally inaccessible stopping times, then $\Pi_p^* A$ is almost surely continuous.*

Proof. First consider the case where $A \in \mathcal{A}^i$. Fix a process $\Pi_p^* A$ satisfying the requirements to be the compensator of A. We will argue that $\Pi_p^* A$ is almost surely continuous. By

Lemma 3.2.4, $\Pi_p^* A \in \mathcal{A}^i$ and $A - \Pi_p^* A \in \mathcal{M}^u$. By Theorem 2.3.9, it holds that $\Pi_p^* A$ almost surely only jumps at predictable times. Therefore, in order to show that $\Pi_p^* A$ is almost surely continuous, it suffices to show that $\Delta \Pi_p^* A_T$ is almost surely zero for all predictable stopping times T. Consider such a stopping time T. Applying Lemma 3.1.8, we find that $E\Delta\Pi_p^* A_T = E\Delta A_T = 0$, since A only jumps at totally inaccessible stopping times. As $\Pi_p^* A$ is increasing, $\Delta\Pi_p^* A_T$ is nonnegative and so we obtain that $\Delta\Pi_p^* A_T$ is almost surely zero, as desired. We conclude that $\Pi_p^* A$ is almost surely continuous.

Next, consider the case where $A \in \mathcal{V}^i$. Define two processes $A_t^+ = \frac{1}{2}((V_A)_t + A_t)$ and $A_t^- = \frac{1}{2}((V_A)_t - A_t)$, by Lemma 1.4.1 we then obtain $A = A^+ - A^-$ where $A^+, A^- \in \mathcal{A}^i$. Furthermore, A^+ and A^- only jump when A does, and so both of these processes only jump at totally inaccessible stopping times. By what we already have shown, $\Pi_p^* A^+$ and $\Pi_p^* A^-$ are almost surely continuous, and so $\Pi_p^* A$ is almost surely continuous as well by Lemma 3.2.4.

Finally, let $A \in \mathcal{V}_\ell^i$. Let (T_n) be a localising sequence such that $A^{T_n} \in \mathcal{V}^i$. By what we already have shown, $\Pi_p^* A^{T_n}$ is almost surely continuous. Applying Lemma 3.2.4, this yields that $\Pi_p^* A$ is almost surely continuous. This concludes the proof. $\square$

For our final basic property of the compensator, we require the following lemma. Note that this result does not follow from Lemma 1.1.13, as the set $[t, \infty)$ is closed.

Lemma 3.2.6. *Let $A \in \mathcal{A}$, let $t \geq 0$ and let $T = \inf\{s \geq 0 \mid A_s \geq t\}$. Then T is a stopping time. If A is predictable, so is T.*

Proof. As A is càdlàg and increasing, we have $(T \leq s) \subseteq (T \leq s, A_T \geq t) \subseteq (A_s \geq t)$. Conversely, we trivially have $(A_s \geq t) \subseteq (T \leq s)$. Therefore, $(T \leq s) = (A_s \geq t) \in \mathcal{F}_s$, so T is a stopping time. In the case where A is predictable, note that

$$[\![T, \infty[\![= \{(s, \omega) \in \mathbb{R}_+ \times \Omega \mid T(\omega) \leq s\} = \{(s, \omega) \in \mathbb{R}_+ \times \Omega \mid A_s(\omega) \geq t\}.$$

As A is predictable, the latter is a predictable set. We conclude that $[\![T, \infty[\![$ is predictable, and Theorem 2.1.12 then shows that T is a predictable stopping time. $\square$

Lemma 3.2.7. *Let $A \in \mathcal{A}$. Assume that A_∞ is square-integrable. Then $\Pi_p^* A_\infty$ is square-integrable as well, and $E(\Pi_p^* A_\infty)^2 \leq 4EA_\infty^2$. If instead $A \in \mathcal{V}$ and $(V_A)_\infty$ is square-integrable, it holds that $\Pi_p^* A_\infty$ is square-integrable and $E(\Pi_p^* A_\infty)^2 \leq 8E(V_A)_\infty^2$.*

Proof. We first consider the case where $A \in \mathcal{A}$ with A_∞ square-integrable. Fix $u \geq 0$. Put $\alpha = \Pi_p^* A_\infty$ and note that

$$\begin{aligned} E(u \wedge \Pi_p^* A_\infty)^2 &= E(u \wedge \alpha)^2 = 2E \int_0^{u \wedge \alpha} (u \wedge \alpha - t) \, dt \\ &= 2E \int_0^{u \wedge \alpha} (u \wedge \alpha - t) 1_{(u \wedge \alpha \geq t)} \, dt \leq 2 \int_0^u E(\alpha - t) 1_{(\alpha \geq t)} \, dt \end{aligned}$$

Now put $T_t = \inf\{s \geq 0 \mid \Pi_p^* A_s \geq t\}$. As $\Pi_p^* A$ is predictable and in $\mathcal{A}$, Lemma 3.2.6 shows that T_t is a predictable stopping time. Also, it holds that $(T_t < \infty) = (\alpha \geq t)$. In particular, on $(\alpha \geq t)$ it holds that $\Pi_p^* A_{T_t-} \leq t$. Letting $M = A - \Pi_p^* A$, we therefore have

$$\begin{aligned} E(\alpha - t) 1_{(\alpha \geq t)} &\leq E(\Pi_p^* A_\infty - \Pi_p^* A_{T_t-}) 1_{(\alpha \geq t)} \\ &= E(A_\infty - A_{T_t-}) 1_{(\alpha \geq t)} - E(M_\infty - M_{T_t-}) 1_{(T_t < \infty)} \\ &\leq E A_\infty 1_{(\alpha \geq t)} - E(M_\infty - M_{T_t-}) 1_{(T_t < \infty)}. \end{aligned}$$

Recalling by Lemma 3.2.4 that $M \in \mathcal{M}^u$, Theorem 1.2.6 and Lemma 3.1.8 yields

$$\begin{aligned} E(M_\infty - M_{T_t-}) 1_{(T_t < \infty)} &= E M_\infty 1_{(T_t < \infty)} - E M_{T_t-} 1_{(T_t < \infty)} \\ &= E 1_{(T_t < \infty)} (M_\infty | \mathcal{F}_{T_t}) - E M_{T_t-} 1_{(T_t < \infty)} \\ &= E M_{T_t} 1_{(T_t < \infty)} - E M_{T_t-} 1_{(T_t < \infty)} \\ &= E \Delta M_{T_t} 1_{(T_t < \infty)} = 0. \end{aligned}$$

Collecting our results, the Cauchy-Schwartz inequality allows us to conclude that

$$\begin{aligned} E(u \wedge \Pi_p^* A_\infty)^2 &\leq 2 \int_0^u E A_\infty 1_{(\alpha > t)} \, dt = 2 E A_\infty (u \wedge \alpha) \\ &= 2 E A_\infty (u \wedge \Pi_p^* A_\infty) \leq 2 (E A_\infty^2)^{1/2} (E(u \wedge \Pi_p^* A_\infty)^2)^{1/2}, \end{aligned}$$

implying $(E(u \wedge \Pi_p^* A_\infty)^2)^{1/2} \leq 2(E A_\infty^2)^{1/2}$ and thus $E(u \wedge \Pi_p^* A_\infty)^2 \leq 4 E A_\infty^2$. Letting u tend to infinity, we obtain by the monotone convergence theorem that $\Pi_p^* A_\infty$ is square-integrable and $E(\Pi_p^* A_\infty)^2 \leq 4 E A_\infty^2$, as desired.

It remains to consider the case where $A \in \mathcal{V}$ and $(V_A)_\infty$ is square-integrable. Define two processes $A_t^+ = \frac{1}{2}((V_A)_t + A_t)$ and $A_t^- = \frac{1}{2}((V_A)_t - A_t)$, by Lemma 1.4.1 we then obtain $A = A^+ - A^-$ where $A^+, A^- \in \mathcal{A}$ and A_∞^+ and A_∞^- are both square-integrable. By what we already have shown, it holds that $E(\Pi_p^* A_\infty^+)^2 \leq 4E(A_\infty^+)^2$ and $E(\Pi_p^* A_\infty^-)^2 \leq 4E(A_\infty^-)^2$. This implies that $\Pi_p^* A_\infty$ is square-integrable, and

$$\begin{aligned} E(\Pi_p^* A_\infty)^2 &= E(\Pi_p^* A_\infty^+ - \Pi_p^* A_\infty^-)^2 \leq E(\Pi_p^* A_\infty^+)^2 + E(\Pi_p^* A_\infty^-)^2 \\ &\leq 4E(A_\infty^+)^2 + 4E(A_\infty^-)^2 \leq 8E(V_A)_\infty^2, \end{aligned}$$

as desired. □

We end the section with some general results on local martingales with paths of finite variation. By $\mathbf{fv}\mathcal{M}_\ell$, we denote the set of local martingales with initial value zero and paths of finite variation. By $\mathbf{iv}\mathcal{M}^u$, we denote the set of uniformly integrable martingales with initial value zero and integrable variation. In other words, $\mathbf{fv}\mathcal{M}_\ell = \mathcal{M}_\ell \cap \mathcal{V}$ and $\mathbf{iv}\mathcal{M}^u = \mathcal{M}^u \cap \mathcal{V}^i$.

Our first lemma shows that every local martingale of finite variation locally is a uniformly integrable martingale of integrable variation.

Lemma 3.2.8. *Let $M \in \mathbf{fv}\mathcal{M}_\ell$. Then there is a localising sequence (T_n) such that M^{T_n} is in $\mathbf{iv}\mathcal{M}^u$.*

Proof. Using Lemma 3.1.5, let (S_n) be a localising sequence such that M^{S_n} is in $\mathcal{M}^u$. Define $R_n = \inf\{t \geq 0 \mid (V_M)_t > n\}$ and put $T_n = S_n \wedge R_n$. Using Lemma 1.2.10, Lemma A.2.8 and Lemma A.2.10, we then obtain

$$(V_{M^{T_n}})_\infty = (V_M)_{T_n} = (V_M)_{T_n-} + \Delta(V_M)_{T_n} \leq n + |\Delta M_{T_n}| \leq 2n + |M_{T_n}|.$$

As $M_{T_n} = M^{S_n}_{R_n}$ and $M^{S_n} \in \mathcal{M}^u$, we find that M_{T_n} is integrable. Therefore, the above shows that M^{T_n} has integrable variation. As we also have $M^{T_n} \in \mathcal{M}^u$, we have obtained a localising sequence (T_n) such that $M^{T_n} \in \mathbf{iv}\mathcal{M}^u$. □

We next apply the compensating projecting to obtain results about $\mathbf{fv}\mathcal{M}_\ell$. Lemma 3.2.9 gives insight into the structure of $\mathbf{fv}\mathcal{M}_\ell$, while Lemma 3.2.10 and Lemma 3.2.11 yields examples of elements of $\mathbf{fv}\mathcal{M}_\ell$ with particular jump structures.

Lemma 3.2.9. *Let $M \in \mathbf{fv}\mathcal{M}_\ell$. Define a process A by putting $A_t = \sum_{0<s\leq t} \Delta M_s$. The sum defining A is absolutely convergent for all $t \geq 0$ and it holds that $A \in \mathcal{V}^i_\ell$. Furthermore, $\Pi^*_p A$ is almost surely continuous and $M = A - \Pi^*_p A$ up to evanescence.*

Proof. As $\sum_{0<s\leq t} |\Delta M_s| \leq (V_M)_t$, it is immediate that the sum defining A is absolutely convergent for all $t \geq 0$. Also, we have $A_t = \sum_{0<s\leq t} \Delta M_s 1_{(\Delta M_s \geq 0)} - \sum_{0<s\leq t} \Delta M_s 1_{(\Delta M_s \leq 0)}$, which shows that A is the difference between two increasing processes. Therefore, A has paths of finite variation. It is immediate that A has càdlàg paths. As the jumps of M may be covered by a sequence of stopping times by Theorem 2.3.8, we obtain that A is adapted, proving that $A \in \mathcal{V}$. It remains to show that A is locally integrable. To this end, we use Lemma 3.2.8 to obtain a localising sequence (T_n) such that $M^{T_n} \in \mathbf{iv}\mathcal{M}^u$. Then

$$|(V_A)_{T_n}| \leq \sum_{0<s<T_n} |\Delta M_s| \leq (V_M)_{T_n} = (V_{M^{T_n}})_\infty,$$

and as the latter is integrable, we conclude that A is locally integrable. Thus, $A \in \mathcal{V}^i_\ell$, in particular the compensator of A is well-defined.

Next, we show that $\Pi^*_p A$ is almost surely continuous. To this end, it suffices by Theorem 2.3.8 to show that $\Delta\Pi^*_p A_T$ is almost surely zero for all stopping times T which are either predictable or totally inaccessible. If T is a totally inaccessible stopping time, it holds by Theorem 2.3.9 that $\Delta\Pi^*_p A_T$ is almost surely zero. Next, let T be a predictable stopping time. Define $N = A - \Pi^*_p A$ and let (T_n) be a localising sequence such that both M^{T_n} and N^{T_n} are in $\mathcal{M}^u$. Note that $N^{T_n} = A^{T_n} - \Pi^*_p A^{T_n}$. As $\Delta\Pi^*_p A^{T_n}_T$ is $\mathcal{F}_{T-}$ measurable by Theorem 2.3.9, we obtain $E(\Delta N^{T_n}_T|\mathcal{F}_{T-}) = E(\Delta A^{T_n}_T|\mathcal{F}_{T-}) - \Delta\Pi^*_p A^{T_n}_T$. Furthermore, $E(\Delta N^{T_n}_T|\mathcal{F}_{T-}) = 0$ by Lemma 3.1.8, yielding $\Delta\Pi^*_p A^{T_n}_T = E(\Delta A^{T_n}_T|\mathcal{F}_{T-}) = E(\Delta M^{T_n}_T|\mathcal{F}_{T-}) = 0$, again by Lemma 3.1.8. Next, note that $\Delta\Pi^*_p A^{T_n}_T = 1_{(T\leq T_n)}\Delta\Pi^*_p A_T$. Letting n tend to infinity, this implies that $\Delta\Pi^*_p A_T$ is almost surely zero. By our earlier deliberations, we may now conclude that $\Pi^*_p A$ is almost surely continuous.

It remains to show that $M = A - \Pi^*_p A$ up to evanescence. However, as $\Delta\Pi^*_p A$ is evanescent by what was already shown, and $\Delta M = \Delta A$, we find that $M - (A - \Pi^*_p A)$ is an element of $\mathbf{fv}\mathcal{M}_\ell$ which is almost surely continuous. Therefore, $M - (A - \Pi^*_p A)$ is evanescent by Theorem 3.1.9, proving that $M = A - \Pi^*_p A$ up to evanescence. □

Lemma 3.2.10. *Let T be a stopping time with $T > 0$. Let $\xi \in \mathcal{L}^1(\mathcal{F}_T)$. Define $A_t = \xi 1_{(t\geq T)}$ and let $M = A - \Pi^*_p A$. It then holds that $M \in \mathcal{M}^u$. If T is predictable and $E(\xi|\mathcal{F}_{T-}) = 0$, then $\Pi^*_p A$ is evanescent and $\Delta M = \Delta A$ almost surely. If T is totally inaccessible, then $\Pi^*_p A$ is almost surely continuous and $\Delta M = \Delta A$ almost surely.*

Proof. That $M \in \mathcal{M}^u$ follows from Theorem 3.2.3, as $A \in \mathcal{V}^i$. Consider the case where T is predictable. We claim that $\Pi^*_p A$ is evanescent. To prove this, let S be any stopping time. As T is predictable, $(S \geq T) \in \mathcal{F}_{(T\wedge S)-} \subseteq \mathcal{F}_{T-}$ by Lemma 2.2.4 and Lemma 2.2.2, so we obtain $EA_S = E\xi 1_{(S\geq T)} = EE(\xi|\mathcal{F}_{T-})1_{(S\geq T)} = 0$, and by Lemma 1.2.8, A is in $\mathcal{M}^u$. Therefore, $\Pi^*_p A$ is evanescent and so $\Delta M = \Delta A$ almost surely. In the case where T is totally inaccessible, Lemma 3.2.5 shows that $\Pi^*_p A$ is almost surely continuous, so $\Delta M = \Delta A$ almost surely in this case as well. □

Lemma 3.2.11. *Let $N \in \mathcal{M}_\ell$ and let T be a stopping time with $T > 0$. Assume that T is predictable or totally inaccesible. Define $A = \Delta N_T 1_{(t\geq T)}$ and put $M = A - \Pi^*_p A$. Then $\Delta M = \Delta A$ almost surely.*

Proof. In the case where T is totally inaccessible, Lemma 3.2.5 implies that $\Pi^*_p A$ is almost surely continuous and so $\Delta M = \Delta A$ almost surely follows immediately. Consider the case where T is predictable. Let (T_n) be a localising sequence such that N^{T_n} is in $\mathcal{M}^u$, and note that

$$A_t^{T_n} = \Delta N_T 1_{(t\wedge T_n \geq T)} = \Delta N_T 1_{(t\geq T)} 1_{(T_n\geq T)} = \Delta N_T^{T_n} 1_{(t\geq T)},$$

where $\Delta N_T^{T_n}$ is integrable by Lemma 3.1.8, and $E(\Delta N_T^{T_n}|\mathcal{F}_{T-}) = 0$. Therefore, Lemma 3.2.10 yields that $\Pi_p^* A^{T_n}$ is almost surely continuous. As $(\Pi_p^* A)^{T_n} = \Pi_p^* A^{T_n}$ by Lemma 3.2.4, we obtain that $\Pi_p^* A$ is almost surely continuous by letting n tend to infinity. As a consequence, $\Delta M = \Delta A$ almost surely in this case as well. □

3.3 The quadratic variation process

In this section, we will prove the existence of the quadratic variation process, and more generally, the quadratic covariation process between two elements of $\mathcal{M}_\ell$. The quadratic covariation process will be a central tool in the construction of the stochastic integral with respect to elements of $\mathcal{M}_\ell$ in Chapter 4.

We begin by proving the following essential result. Recall that $\mathbf{fv}\mathcal{M}_\ell$ denotes the subspace of elements of $\mathcal{M}_\ell$ with paths of finite variation. We also introduce $\mathcal{M}_\ell^b$ as the subspace of $\mathcal{M}_\ell$ such that there is a localising sequence (T_n) with the property that M^{T_n} is bounded. We refer to $\mathcal{M}_\ell^b$ as the space of locally bounded local martingales.

Theorem 3.3.1 (Fundamental theorem of local martingales)**.** *Let $M \in \mathcal{M}_\ell$. There exists $M^b \in \mathcal{M}_\ell^b$ and $M^v \in \mathbf{fv}\mathcal{M}_\ell$ such that $M = M^b + M^v$ almost surely.*

Proof. Define $A_t = \sum_{0<s\leq t} \Delta M_s 1_{(|\Delta M_s|>\varepsilon)}$. By Lemma A.2.3, M has only finitely many jumps larger or equal to ε on any finite interval, yielding that A is a well-defined càdlàg process. As $(V_A)_t = \sum_{0<s\leq t} |\Delta M_s 1_{(|\Delta M_s|>\varepsilon)}| \leq \sum_{0<s\leq t} |\Delta M_s|$, we obtain by Lemma 3.2.9 that $A \in \mathcal{V}_\ell^i$. In particular, the compensator of A is well-defined.

Now put $M^v = A - \Pi_p^* A$, then $M^v \in \mathbf{fv}\mathcal{M}_\ell$ and we have $M = (M - M^v) + M^v$, where M^v and $M - M^v$ are both in $\mathcal{M}_\ell$. We will argue that there is a localising sequence (T_n) such that $(M - M^v)^{T_n}$ almost surely is bounded. To this end, let (S_n) be a localising sequence such that $M^{S_n} \in \mathcal{M}^u$ and $A^{S_n} \in \mathcal{V}^i$. We claim that $(M - M^v)^{S_n}$ almost surely has bounded jumps. To show this, let T be some stopping time. Note that

$$|\Delta(M - A)_T| = |\Delta M_T - \Delta M_T 1_{(|\Delta M_T|>\varepsilon)}| = |\Delta M_T 1_{(|\Delta M_T|\leq\varepsilon)}| \leq \varepsilon.$$

Also note that $\Delta M_T^{S_n} = \Delta M_T 1_{(T\le S_n)}$ and analogously for other càdlàg processes. Therefore, if T is totally inaccessible, Theorem 2.3.9 yields that, almost surely,

$$\begin{aligned}|\Delta(M-M^v)_T^{S_n}| &= |\Delta(M-M^v)_T 1_{(T\le S_n)}| = |\Delta M_T - (\Delta A_T - \Delta(\Pi_p^* A)_T)|1_{(T\le S_n)}\\ &= |\Delta M_T - \Delta A_T|1_{(T\le S_n)} \le \varepsilon.\end{aligned}$$

Next, let T be predictable. We have $A^{S_n}\in\mathcal{V}^i$, and so Lemma 3.2.4 yields $\Pi_p^* A^{S_n}\in\mathcal{V}^i$, $(M^v)^{S_n} = A^{S_n} - \Pi_p^* A^{S_n}$ and $(M^v)^{S_n}\in\mathcal{M}^u$. In particular, by Lemma 3.1.8, $(M^v)_T^{S_n}$ is integrable with $E(\Delta(M^v)_T^{S_n}|\mathcal{F}_{T-}) = 0$. As A_T and $\Pi_p^* A_T$ both are integrable as well, we conclude $0 = E(\Delta(M^v)_T^{S_n}|\mathcal{F}_{T-}) = E(\Delta A_T^{S_n} - \Delta\Pi_p^* A_T^{S_n}|\mathcal{F}_{T-}) = E(\Delta A_T^{S_n}|\mathcal{F}_{T-}) - \Delta\Pi_p^* A_T^{S_n}$ by Theorem 2.3.9. Thus, $E(\Delta A_T^{S_n}|\mathcal{F}_{T-}) = \Delta\Pi_p^* A_T^{S_n}$. As $M^{S_n}\in\mathcal{M}^u$, Lemma 3.1.8 shows that $\Delta M_T^{S_n}$ is also integrable and $E(\Delta M_T^{S_n}|\mathcal{F}_{T-}) = 0$, so that

$$\begin{aligned}\Delta((M-M^v)^{S_n})_T &= \Delta M_T^{S_n} - (\Delta A_T^{S_n} - \Delta\Pi_p^* A_T^{S_n})\\ &= \Delta M_T^{S_n} - \Delta A_T^{S_n} + E(\Delta A_T^{S_n}|\mathcal{F}_{T-})\\ &= \Delta M_T^{S_n} - \Delta A_T^{S_n} - E(\Delta M_T^{S_n} - \Delta A_T^{S_n}|\mathcal{F}_{T-})\\ &= \Delta(M-A)_T 1_{(T\le S_n)} - E(\Delta(M-A)_T 1_{(T\le S_n)}|\mathcal{F}_{T-}),\end{aligned}$$

which yields $|\Delta((M-M^v)^{S_n})_T|\le 2\varepsilon$ almost surely. We have now shown that for any stopping time T which is predictable or totally inaccessible, $|\Delta((M-M^v)^{S_n})_T|\le 2\varepsilon$ almost surely. As the jump times of $(M-M^v)^{S_n}$ by Theorem 2.3.8 can be covered by a regular sequence of stopping times, this implies that $|\Delta((M-M^v)^{S_n})|\le 2\varepsilon$ almost surely. Letting n tend to infinity, we obtain that $M-M^v$ almost surely has jumps bounded by 2ε. Now let M^b be a modification of $M-M^v$ in $\mathcal{M}_\ell$ with jumps bounded by 2ε. Defining a sequence (T_n) by putting $T_n = \inf\{t\ge 0 \mid |M_t^b| > n\}$, we obtain by Lemma 1.1.13 that (T_n) is a localising sequence, and by the boundedness of the jumps, $(M^b)^{T_n}$ is in $\mathcal{M}^b$. Thus, $M^b\in\mathcal{M}_\ell^b$. As $M = M^b + M^v$ almost surely, the proof is complete. □

Theorem 3.3.1 will be essential to our construction of both the quadratic variation process for local martingales and to our construction of the stochastic integral. Now, recall that we in Theorem 1.3.6 proved the existence of the quadratic variation process for bounded martingales. Our next objective is to extend this result to all local martingales. To obtain this, we require two preliminary results. For the first result, recall from Section 1.4 that we for a progressive and almost surely integrable process H and an element $A\in\mathcal{V}$ ensured the existence of a process $H\cdot A\in\mathcal{V}$ such that almost surely, for all $t\ge 0$ $(H\cdot A)_t$ is equal to the Lebesgue integral of H with respect to A over $[0,t]$.

Theorem 3.3.2. *Let $M\in\mathbf{fv}\mathcal{M}_\ell$ and let X be a predictable process. Assume that there is a localising sequence (T_n) such that $X^{T_n}1_{(T_n>0)}$ is bounded. X is then almost surely integrable* *with respect to M, and $X \cdot M$ is in $\mathcal{M}_\ell$. The existence of the localising sequence holds in particular if X is càglàd, adapted and has initial value zero.*

Proof. First assume that X is càglàd, adapted and has initial value zero. In this case, we may put $T_n = \inf\{t \geq 0 \mid |X_t| > n\}$ and obtain that (T_n) is a localising sequence such that X^{T_n} is bounded, in particular $X^{T_n} 1_{(T_n > 0)}$ is bounded. This proves the final claim of the theorem.

Now consider the case where $M \in \mathbf{iv}\mathcal{M}^u$ and assume that X is predictable and that there is a localising sequence (T_n) such that $X^{T_n} 1_{(T_n>0)}$ is bounded. As $(V_M)_\infty$ is integrable in this case, it is almost surely finite. Note that for fixed $t \geq 0$ and ω and n large enough, it holds that $X^{T_n}_s(\omega) 1_{(T_n(\omega)>0)} = X_s(\omega)$ for $0 \leq s \leq t$. Therefore, X is almost surely integrable with respect to M. In particular, by Theorem 1.4.3, the integral process $X \cdot M$ is uniquely defined up to indistinguishability. By taking a modification of M, we may assume that $(V_M)_\infty$ only takes finite values and retain the property that X is almost surely integrable with respect to M as well as retain the process $X \cdot M$.

We wish to prove that $X \cdot M$ is in $\mathcal{M}_\ell$. To this end, let ν_ω be the measure induced by $M(\omega)$ on $(\mathbb{R}_+, \mathcal{B}_+)$ according to Theorem A.2.9. By Lemma 1.4.2, we find that $(\nu_\omega)_{\omega \in \Omega}$ is a P-integrable $(\Omega, \mathcal{F})$ kernel on $(\mathbb{R}_+, \mathcal{B}_+)$. Theorem A.1.13 therefore yields the existence of a unique signed measure μ on $\mathcal{B}_+ \otimes \mathcal{F}$ such that for any $A \in \mathcal{B}_+$ and $F \in \mathcal{F}$, it holds that $\mu(A \times F) = \int_F \nu_\omega(A) \, \mathrm{d}P(\omega)$. In order to obtain that $X \cdot M$ is in $\mathcal{M}_\ell$, we now proceed in three steps.

Step 1. Proof that μ is zero on Σ^p. First, we argue that μ is zero on Σ^p. To this end, let $\mathcal{H} = \{A \in \Sigma^p \mid \mu(A) = 0\}$. It then holds that $\mathcal{H}$ is a Dynkin class. In order to show the result, it therefore suffices to show that $\mathcal{H}$ contains a generator for Σ^p which is stable under taking intersections. By Lemma 2.1.6, Σ^p is generated by $\{[\![T, \infty[\![\mid T \in \mathcal{T}_p\}$, where $\mathcal{T}_p$ denotes the set of predictable stopping times. Furthermore, this generating class is stable under taking intersections. Therefore, by Lemma A.1.1, in order to show that μ is zero on Σ^p, it suffices to show that $\mu([\![T, \infty[\![)$ is zero for all predictable stopping times T. Let T be such a predictable stopping time. By Theorem A.1.17, we have

$$\mu([\![T, \infty[\![) = \int 1_{[\![T,\infty[\![} \, \mathrm{d}\mu = \int \int 1_{(T \leq t)}(\omega) \, \mathrm{d}\nu_\omega(t) \, \mathrm{d}P(\omega) = E 1_{(T<\infty)}(M_\infty - M_{T-}).$$

Now, by Lemma 3.1.8, we know that ΔM_T is integrable and $E(\Delta M_T | \mathcal{F}_{T-}) = 0$. As it holds that $(T < \infty) \in \mathcal{F}_{T-}$, this yields $E(\Delta M_T 1_{(T<\infty)} | \mathcal{F}_{T-}) = 0$ as well. And as M_T is integrable, we obtain in particular that $M_T 1_{(T<\infty)}$ and thus $M_{T-} 1_{(T<\infty)}$ are integrable. Noting that

$M_{T-}1_{(T<\infty)}$ is $\mathcal{F}_{T-}$ measurable by Lemma 2.2.7, we get $E(M_T 1_{(T<\infty)}|\mathcal{F}_{T-}) = M_{T-}1_{(T<\infty)}$ and so $EM_T 1_{(T<\infty)} = EM_{T-}1_{(T<\infty)}$. Thus,

$$E1_{(T<\infty)}(M_\infty - M_{T-}) = E1_{(T<\infty)}(M_\infty - M_T) = E(M_\infty - M_T) = 0.$$

Collecting our conclusions, we have now shown that μ is zero on Σ^p.

Step 2. Proof that $Y \cdot M$ is in $\mathcal{M}^u$ for particular Y. Now let Y be any predictable process which is integrable with respect to μ, this is well-defined as we know by Lemma 1.1.6 and Lemma 1.1.8 that $\Sigma^p \subseteq \Sigma^\pi \subseteq \mathcal{B}_+ \otimes \mathcal{F}$. Invoking Theorem A.1.17, we then find that $Y(\omega)$ is integrable with respect to $M(\omega)$ over $\mathbb{R}_+$ for P almost all ω, that the result is integrable with respect to P, and $E\int_0^\infty Y_t \,\mathrm{d}M_t = \int\int Y_t(\omega)\,\mathrm{d}\nu_\omega(t)\,\mathrm{d}P(\omega) = \int Y\,\mathrm{d}\mu$, and this latter expression is zero by what we already have shown. Now, as M was arbitrary in $\mathbf{iv}\mathcal{M}^u$, this also holds for M^T, where T is any stopping time. Therefore, we obtain for any T that $E(Y\cdot M)_T = E(Y\cdot M^T)_\infty = 0$. By Lemma 1.2.8, this implies that $Y \cdot M$ is in $\mathcal{M}^u$. This holds for any predictable Y which is integrable with respect to μ.

Step 3. Proof that $X \cdot M$ is in $\mathcal{M}_\ell$. Finally, as $X^{T_n}1_{(T_n>0)}$ is predictable and bounded, it is integrable with respect to μ. As M^{T_n} is in $\mathbf{iv}\mathcal{M}^u$ whenever $M \in \mathbf{iv}\mathcal{M}^u$, our previous step shows that $X^{T_n}1_{(T_n>0)} \cdot M^{T_n}$ is in $\mathcal{M}^u$. By the properties of Lebesgue integration, we have

$$(X\cdot M)^{T_n} = 1_{(T_n>0)}(X\cdot M)^{T_n} = 1_{(T_n>0)}(X^{T_n}\cdot M^{T_n}) = X^{T_n}1_{(T_n>0)}\cdot M^{T_n}$$

almost surely. Thus, $X \cdot M$ is in $\mathcal{M}_\ell$, as desired.

It now only remains to extend our results to the case where M is in $\mathbf{fv}\mathcal{M}_\ell$ instead of $\mathbf{iv}\mathcal{M}^u$. Assume that $M \in \mathbf{fv}\mathcal{M}_\ell$. By Lemma 3.2.8, there is a localising sequence (T_n) such that $M^{T_n} \in \mathbf{iv}\mathcal{M}^u$. By what we already have shown, X is integrable with respect to M^{T_n}, and $X \cdot M^{T_n}$ is in $\mathcal{M}_\ell$. Therefore, X is integrable with respect to M as well, and we obtain $(X\cdot M)^{T_n} = X \cdot M^{T_n}$. Thus, $(X\cdot M)^{T_n}$ is in $\mathcal{M}_\ell$ for all $n \geq 1$, and so Lemma 3.1.7 shows that $X\cdot M$ is in $\mathcal{M}_\ell$, as desired. □

Lemma 3.3.3. *Let $A \in \mathcal{V}^i$ and let $M \in \mathcal{M}^b$. Then $M_tA_t - \int_0^t M_s\,\mathrm{d}A_s$ is in $\mathcal{M}^u$.*

Proof. Note that the process $\int_0^t M_s\,\mathrm{d}A_s$ is always a well-defined element of $\mathcal{V}$ by Theorem 1.4.3, so the conclusion is well-formed. First assume that $A \in \mathcal{A}^i$. Let $c \geq 0$ be such that $|M_t| \leq c$ for $t \geq 0$. We will apply Lemma 1.2.8 to obtain the result. To this end, first note that by Lemma A.2.12, we have

$$E(V_{M\cdot A})_\infty = \lim_{t\to\infty} E(V_{M\cdot A})_t \leq \lim_{t\to\infty} E\int_0^t |M_s|\,\mathrm{d}A_s \leq cEA_\infty,$$

which is finite, so $M \cdot A \in \mathcal{V}^i$, in particular $(M \cdot A)$ is almost surely convergent. Also, as M is bounded, M is almost surely convergent, and as A is integrable and increasing, A is almost surely convergent as well. Therefore, the process $M_t A_t - \int_0^t M_t \, \mathrm{d}A_t$ is almost surely convergent, and so Lemma 1.2.8 yields that in order to prove that the process is in $\mathcal{M}^u$, it suffices to show that for any stopping time T, $M_T A_T - \int_0^T M_t \, \mathrm{d}A_t$ is integrable and has mean zero.

Fix such a stopping time T. As we have

$$M_T A_T - \int_0^T M_t \, \mathrm{d}A_t = M_\infty^T A_\infty^T - \int_0^\infty M_t 1_{(t \le T)} \, \mathrm{d}A_t^T = M_\infty^T A_\infty^T - \int_0^\infty M_t^T \, \mathrm{d}A_t^T,$$

we find that it suffices to prove that when $M \in \mathcal{M}^b$ and $A \in \mathcal{A}^i$, $M_\infty A_\infty - \int_0^\infty M_t \, \mathrm{d}A_t$ is integrable and has mean zero. Integrability follows since $E|M_\infty A_\infty| \le cEA_\infty$, which is finite, and $E|\int_0^\infty M_t \, \mathrm{d}A_t| \le cEA_\infty$ as well, which is also finite. It remains to show that the expectation is zero. To this end, define $T_t = \inf\{s \ge 0 | A_s \ge t\}$. By Lemma 3.2.6, T_t is a stopping time. In particular, as $(T_t < \infty)$ is in $\mathcal{F}_{T_t}$, we have $EM_{T_t} 1_{(T_t < \infty)} = EM_\infty 1_{(T_t < \infty)}$ and so, applying Lemma A.2.14 twice, we find

$$\begin{aligned} E \int_0^\infty M_t \, \mathrm{d}A_t &= E \int_0^\infty M_{T_t} 1_{(T_t < \infty)} \, \mathrm{d}t = \int_0^\infty EM_{T_t} 1_{(T_t < \infty)} \, \mathrm{d}t \\ &= \int_0^\infty EM_\infty 1_{(T_t < \infty)} \, \mathrm{d}t = E \int_0^\infty M_\infty \, \mathrm{d}A_t = EM_\infty A_\infty. \end{aligned}$$

This concludes the proof for the case $A \in \mathcal{A}^i$. Now assume $A \in \mathcal{V}^i$. By Lemma 1.4.1, there is a decomposition $A = A^+ - A^-$, where $A^+, A^- \in \mathcal{A}^i$. As

$$M_t A_t - \int_0^t M_s \, \mathrm{d}A_s = M_t A_t^+ - \int_0^t M_s \, \mathrm{d}A_s^+ - \left(M_t A_t^- - \int_0^t M_s \, \mathrm{d}A_s^- \right),$$

the general result follows. □

Theorem 3.3.4. *Let $M \in \mathcal{M}_\ell$. There exists a process $[M] \in \mathcal{A}$, unique up to indistinguishability, such that $M^2 - [M] \in \mathcal{M}_\ell$ and $\Delta[M] = (\Delta M)^2$ almost surely. If $M, N \in \mathcal{M}_\ell$, there exists a process $[M, N] \in \mathcal{V}$, unique up to indistinguishability, such that $MN - [M, N] \in \mathcal{M}_\ell$ and $\Delta[M, N] = \Delta M \Delta N$ almost surely. We refer to $[M]$ as the quadratic variation of M, and we refer to $[M, N]$ as the quadratic covariation of M and N.*

Proof. We begin by proving existence and uniqueness of the quadratic variation, the existence and uniqueness of the quadratic covariation will follow by a simple polarization argument.

We first consider uniqueness. If A and B are two processes in $\mathcal{A}$ such that both $M^2 - A$ and $M^2 - B$ are in $\mathcal{M}_\ell$ with $\Delta A = \Delta B = (\Delta M)^2$ almost surely, we obtain that $A - B$ is in $\mathcal{M}_\ell$, is almost surely continuous and has paths of finite variation. Therefore, A and B are indistinguishable by Theorem 3.1.9.

Next, we consider existence. We first consider the case where $M = M^b + M^i$, with $M^b \in \mathcal{M}^b$ and $M^i \in \mathbf{iv}\mathcal{M}^u$. By Lemma A.2.16, $\sum_{0<s\le t}(\Delta M^i_t)^2$ is absolutely convergent for any $t \ge 0$, and we may therefore define a process A^i in $\mathcal{A}$ by putting $A^i_t = \sum_{0<s\le t}(\Delta M^i_t)^2$. As M^b is bounded, Lemma A.2.16 shows that $\sum_{0<s\le t}\Delta M^b_t \Delta M^i_t$ is almost surely absolutely convergent, and so we may define a process A^x in $\mathcal{V}$ by putting $A^x_t = \sum_{0<s\le t}\Delta M^b_t \Delta M^i_t$. Finally, by Theorem 1.3.6, there exists a process $[M^b]$ in $\mathcal{A}^i$ such that $(M^b)^2 - [M^b] \in \mathcal{M}_\ell$ and $\Delta[M^b] = (\Delta M^b)^2$. We put $A_t = [M^b]_t + 2A^x + A^i$ and claim that A satisfies the criteria of the theorem.

To this end, first note that A clearly is càdlàg adapted of finite variation, and for $0 \le s \le t$, we have $[M^b]_t \ge [M^b]_s + \sum_{s<u\le t}(\Delta M^b_u)^2$ almost surely, and so

$$\begin{aligned} A_t - A_s &= [M^b]_t - [M^b]_s + 2(A^x_t - A^x_s) + A^i_t - A^i_s \\ &\ge \sum_{s<u\le t}(\Delta M^b_u)^2 + 2\Delta M^b_u \Delta M^i_u + (\Delta M^i_u)^2 = \sum_{s<u\le t}(\Delta M^b_u + \Delta M^i_u)^2 \ge 0, \end{aligned}$$

almost surely, showing that A is almost surely increasing. To prove that $M^2 - A$ is in $\mathcal{M}_\ell$, note that $M^2 - A = (M^b)^2 - [M^b] + 2(M^b M^i - A^x) + (M^i)^2 - A^i$. Here, $(M^b)^2 - [M^b]$ is in $\mathcal{M}^2$ by Theorem 1.3.6, in particular in $\mathcal{M}_\ell$. By the integration-by-parts formula, we have $(M^i)^2_t - A^i_t = 2\int_0^t M^i_{s-}\,\mathrm{d}M^i_s$, where the integral is well-defined as M_{s-} is bounded on compacts. By Lemma 3.3.2, this process is a local martingale, and so $(M^i)^2 - A^i$ is in $\mathcal{M}_\ell$. Thus, in order to obtain that $M^2 - A$ is in $\mathcal{M}_\ell$, it suffices to show that $M^b M^i - A^x$ is in $\mathcal{M}_\ell$. By Lemma 3.3.3, $M^b_t M^i_t - \int_0^t M^b_s\,\mathrm{d}M^i_s$ is in $\mathcal{M}_\ell$, so it suffices to show that $\int_0^t M^b_s\,\mathrm{d}M^i_s - A^x_t$ is in $\mathcal{M}_\ell$. As ΔM^b is bounded, it is integrable, and so we have

$$\int_0^t M^b_s\,\mathrm{d}M^i_s = \int_0^t \Delta M^b_s\,\mathrm{d}M^i_s + \int_0^t M^b_{s-}\,\mathrm{d}M^i_s = A^x_t + (M^b_- \cdot M^i)_t.$$

As $M^b_- \cdot M^i$ is in $\mathcal{M}_\ell$ by Lemma 3.3.2, we finally conclude that $M^b M^i - A^x$ is in $\mathcal{M}_\ell$. Thus, $M^2 - A$ is in $\mathcal{M}_\ell$. As it is immediate that $\Delta A_t = \Delta[M^b]_t + 2\Delta M^b_t \Delta M^i_t + (\Delta M^i_t)^2 = (\Delta M_t)^2$ almost surely, this proves existence in the case where $M = M^b + M^i$, where $M^b \in \mathcal{M}^b$ and $M^i \in \mathbf{iv}\mathcal{M}^u$.

Now consider the case of a general $M \in \mathcal{M}_\ell$. By Theorem 3.3.1, there exists $M^b \in \mathcal{M}^b_\ell$ and $M^i \in \mathbf{fv}\mathcal{M}_\ell$ such that $M = M^b + M^i$ almost surely. Put $N = M^b + M^i$ and let (T_n) be a localising sequence such that $(M^b)^{T_n}$ is in $\mathcal{M}^b$ and $(M^i)^{T_n}$ is in $\mathbf{iv}\mathcal{M}^u$. By what was already shown, there exists a process $[N^{T_n}] \in \mathcal{A}$, unique up to indistinguishability, such that $(N^{T_n})^2 - [N^{T_n}]$ is in $\mathcal{M}_\ell$ and $\Delta[N^{T_n}] = (\Delta N^{T_n})^2$ almost surely. By uniqueness, we have $[N^{T_{n+1}}]^{T_n} = [N^{T_n}]$ up to indistinguishability. Therefore, these processes may be pasted together to yield a process $[N] \in \mathcal{A}$ such that $\Delta[N] = (\Delta N)^2$ almost surely and $N^2 - [N] \in \mathcal{M}_\ell$. As N and M are indistinguishable, $[N]$ also satifies the criteria for being the quadratic variation of M. This concludes the proof of existence.

Considering the quadratic covariation, let $M, N \in \mathcal{M}_\ell$ be given. Recalling the polarization identity $4xy = (x+y)^2 - (x-y)^2$ for $x, y \in \mathbb{R}$, we define $[M,N] = \frac{1}{4}([M+N] - [M-N])$. We then obtain $MN - [M,N] = \frac{1}{4}((M+N)^2 - [M+N]) - \frac{1}{4}((M-N)^2 - [M-N])$. This shows that $MN - [M,N]$ is a local martingale. Furthermore, we have $[M,N] \in \mathcal{V}$, and $\Delta[M,N] = \frac{1}{4}((\Delta(M+N))^2 - (\Delta(M-N))^2) = \Delta M \Delta N$ almost surely. This proves existence of the quadratic covariation. Uniqueness follows as for the quadratic variation. □

Theorem 3.3.4 yields the existence and uniqueness of the quadratic variation and quadratic covariation processes for local martingales, and is one of the main results of this section. A useful consequence of the result is the following.

Lemma 3.3.5. *Let $M \in \mathcal{M}_\ell$. Then it almost surely holds that for all $t \geq 0$, $\sum_{0<s\leq t}(\Delta M_s)^2$ is absolutely convergent.*

Proof. By Theorem 3.3.4, we know that there exists a process $[M] \in \mathcal{A}$ with the property that $\Delta[M]_s = (\Delta M_s)^2$ almost surely. As we then have $\sum_{0<s\leq t}(\Delta M_s)^2 \leq [M]_t$ for all $t \geq 0$ almost surely, the result follows. □

In the remainder of this section, we investigate the fundamental properties of the quadratic covariation and how the quadratic covariation may be applied to understand the structure of local martingales. We first calculate the quadratic variation and quadratic covariation for the most commonly occurring process, the Brownian motion. Afterwards, we work towards proving some general properties of the quadratic covariation.

Theorem 3.3.6. *Let W be a p-dimensional $\mathcal{F}_t$ Brownian motion. For $i \leq p$, $[W^i]_t = t$ up to indistinguishability, and for $i, j \leq p$ with $i \neq j$, $[W^i, W^j]$ is zero up to indistinguishability.*

Proof. By Theorem 1.2.13, it holds for $i \leq p$ that $(W^i_t)^2 - t$ is a martingale, in particular an element of $\mathbf{c}\mathcal{M}_\ell$, and so $[W^i]_t = t$ up to indistinguishability, by the characterization given in Theorem 3.3.4. Likewise, Theorem 1.2.13 shows that for $i, j \leq p$ with $i \neq j$, $W^i_t W^j_t$ is a martingale, in particular an element of $\mathbf{c}\mathcal{M}_\ell$, so $[W^i, W^j]$ is zero up to indistinguishability by Theorem 3.3.4. □

Lemma 3.3.7. *Let $M \in \mathcal{M}_\ell$, let T be a stopping time and let ξ be $\mathcal{F}_T$ measurable. The process $\xi(M - M^T)$ is in $\mathcal{M}_\ell$.*

Proof. First consider the case where ξ is bounded and $M \in \mathcal{M}^u$. With S being some stopping time, we obtain that $\xi 1_{(T\leq S)}$ is $\mathcal{F}_{S\wedge T}$ measurable by Lemma 1.1.11, and thus

$$E\xi(M_S - M_S^T) = E\xi 1_{(T\leq S)}(M_S - M_S^T) = E\xi 1_{(T\leq S)} E(M_S - M_{S\wedge T}|\mathcal{F}_{S\wedge T}) = 0,$$

by Theorem 1.2.6. By Lemma 1.2.8, $\xi(M - M^T)$ is in $\mathcal{M}^u$. Next, consider the general case where ξ is merely $\mathcal{F}_T$ measurable and $M \in \mathcal{M}_\ell$. Let (R_n) be a localising sequence such that $M^{R_n} \in \mathcal{M}^u$, and define $S_n = T_{(|\xi|>n)}$. Put $T_n = S_n \wedge R_n$. We then obtain

$$(\xi(M - M^T))^{T_n} = \xi(M^{T_n} - M^{T\wedge T_n}) = \xi 1_{(|\xi|\leq n)}(M^{R_n} - (M^{R_n})^T),$$

which is in $\mathcal{M}^u$ by what was already shown. Thus, $\xi(M - M^T)$ is in $\mathcal{M}_\ell$. □

Lemma 3.3.8. *Let M and N be in $\mathcal{M}_\ell$, and let T by any stopping time. The quadratic covariation satisfies the following properties up to indistinguishability.*

(1). $[M, M] = [M]$.

(2). $[\cdot, \cdot]$ *is symmetric and linear in both of its arguments.*

(3). For any $\alpha \in \mathbb{R}$, $[\alpha M] = \alpha^2[M]$.

(4). It holds that $[M + N] = [M] + 2[M, N] + [N]$.

(5). It holds that $[M, N]^T = [M^T, N] = [M, N^T] = [M^T, N^T]$.

(6). $[M, N]$ *is zero if and only if* $MN \in \mathcal{M}_\ell$.

(7). M *is evanescent if and only if* $[M]$ *is evanescent.*

(8). M *is evanescent if and only if* $[M, N]$ *is zero for all* $N \in \mathcal{M}_\ell$.

(9). If $F \in \mathcal{F}_0$, $1_F[M, N] = [1_F M, N] = [M, 1_F N] = [1_F M, 1_F N]$.

Proof. **Proof of (1).** We know that $[M]$ is in $\mathcal{A}$ and satisfies $M^2 - [M] \in \mathcal{M}_\ell$. Therefore, $[M]$ is in particular in $\mathcal{V}$, and therefore satisfies the requirements characterizing $[M, M]$. By uniqueness, we conclude that $[M, M] = [M]$ up to indistinguishability.

Proof of (2). As $MN - [M, N]$ is a uniformly integrable martingale if and only if this holds for $NM - [M, N]$, we have by uniqueness that the quadratic covariation is symmetric in the sense that $[M, N] = [N, M]$ up to indistinguishability. In particular, it suffices to prove that the quadratic covariation is linear in its first coordinate. Fix $M, M' \in \mathcal{M}_\ell$ and $\alpha, \beta \in \mathbb{R}$, then $(\alpha M + \beta M')N - (\alpha[M, N] + \beta[M', N]) = \alpha(MN - [M, N]) + \beta(M'N - [M', N])$, so $(\alpha M + \beta M')N - (\alpha[M, N] + \beta[M', N])$ is in $\mathcal{M}_\ell$ and so by uniqueness, we have the linearity relationship $[\alpha M + \beta M', N] = \alpha[M, N] + \beta[M', N]$ up to indistinguishability.

Proof of (3). This is immediate from $[\alpha M] = [\alpha M, \alpha M] = \alpha^2[M, M] = \alpha^2[M]$, using the linearity properties already proven.

Proof of (4). This follows as

$$\begin{aligned} [M + N] &= [M + N, M + N] \\ &= [M, M] + [M, N] + [N, M] + [N, N] \\ &= [M] + 2[M, N] + [N], \end{aligned}$$

using the symmetry and linearity properties already proven.

Proof of (5). Note that as M^T and N^T are in $\mathcal{M}_\ell$, the conclusion is well-defined by Lemma 3.1.3. To prove the result, first note that by symmetry, it suffices to prove $[M, N]^T = [M^T, N]$, and this will be accomplished if we can show that $M^T N - [M, N]^T$ is in $\mathcal{M}_\ell$. Note that

$$\begin{aligned} M^T N - [M, N]^T &= (MN - [M, N])^T + M^T(N - N^T) \\ &= (MN - [M, N])^T + M_T(N - N^T), \end{aligned}$$

where $(MN - [M, N])^T \in \mathcal{M}_\ell$ by Lemma 3.1.3. By Lemma 3.3.7, $M_T(N - N^T)$ is in $\mathcal{M}_\ell$ as well. The result follows.

Proof of (6). This is immediate from the definition of the quadratic covariation.

Proof of (7). If M is the zero process, then the zero process satisfies the requirements for being the quadratic variation of M. Conversely, assume that $[M]$ is evanescent. Then M^2 is in $\mathcal{M}_\ell$. Letting T_n be a localising sequence for M^2 such that $(M^2)^{T_n}$ is in $\mathcal{M}$, we find that $EM^2_{T_n \wedge t} = E(M^2)^{T_n}_t = 0$, so that $M^2_{T_n \wedge t}$ is almost surely zero. Therefore, M_t is almost surely zero as well. As $t \geq 0$ was arbitrary and M is càdlàg, we conclude that M is evanescent.

Proof of (8). Assume that M is evanescent. Then the zero process satisfies the requirements characterizing $[M,N]$ for all $N \in \mathcal{M}_\ell$, and so $[M,N]$ is evanescent for all $N \in \mathcal{M}_\ell$. Conversely, assume that $[M,N]$ is evanescent for all $N \in \mathcal{M}_\ell$. In particular, $[M,M]$ is evanescent, so by what was already shown, M is evanescent.

Proof of (9). Note that the conclusion is well-defined, as $1_F M$ is in $\mathcal{M}_\ell$ by Lemma 3.1.3. By the properties already proven for the quadratic covariation, it suffices to prove that for any $F \in \mathcal{F}_0$ and $M,N \in \mathcal{M}_\ell$, $1_F[M,N] = [1_F M, N]$. However, we know that $MN - [M,N]$ is in $\mathcal{M}_\ell$, and so by Lemma 3.1.3, $1_F MN - 1_F[M,N]$ is in $\mathcal{M}_\ell$. Therefore, by the characterisation of the quadratic covariation, $1_F[M,N]$ is the quadratic covariation process of $1_F M$ and N, meaning that $1_F[M,N] = [1_F M, N]$, as desired. □

For the next result, recall that integrals of the form $\int_0^t h(s)|\,\mathrm{d}f_s|$ denote integration with respect to the variation of f.

Theorem 3.3.9 (Kunita-Watanabe)**.** *Let $M,N \in \mathcal{M}_\ell$, and let H and K be measurable processes. Then it almost surely holds that for all $t \geq 0$,*

$$\int_0^\infty |H_t K_t||\,\mathrm{d}[M,N]_t| \leq \left(\int_0^\infty H_t^2 \,\mathrm{d}[M]_t\right)^{\frac{1}{2}} \left(\int_0^\infty K_t^2 \,\mathrm{d}[N]_t\right)^{\frac{1}{2}}.$$

Proof. First note that the result is well-defined for each ω, as $[M,N]$, $[M]$ and $[N]$ have paths of finite variation for each ω, and the mappings $|H_t K_t|$, H_t^2 and K_t^2 from $\mathbb{R}_+$ to $\mathbb{R}$ are Borel measurable for each ω.

Applying Lemma A.2.18, it suffices to prove that almost surely, it holds that for all $0 \leq s \leq t$, $|[M,N]_t - [M,N]_s| \leq \sqrt{[M]_t - [M]_s}\sqrt{[N]_t - [N]_s}$. As the processes are càdlàg, it suffices to prove the result almost surely for any pair of rational s and t. Fix any such pair, by Lemma A.2.17 it suffices to prove that $\lambda^2([M]_t - [M]_s) + 2\lambda([M,N]_t - [M,N]_s) + [N]_t - [N]_s \geq 0$ for all $\lambda \in \mathbb{Q}$. Thus, we need to prove that this inequality holds almost surely for rational s, t and λ with $0 \leq s \leq t$. Note that $\lambda^2[M]_s + 2\lambda[M,N]_s + [N]_s = [\lambda M]_s + 2[\lambda M, N]_s + [N]_s = [\lambda M + N]_s$, and $[\lambda M + N]_s \leq [\lambda M + N]_t$, so by performing the same calculations in reverse, we obtain $\lambda^2[M]_s + 2\lambda[M,N]_s + [N]_s \leq \lambda^2[M]_t + 2\lambda[M,N]_t + [N]_t$, yielding the desired conclusion. The theorem now follows from Lemma A.2.18. □

We end the section with two results describing the interplay between the quadratic variation and $\mathcal{M}^2$.

Theorem 3.3.10. *Let $M \in \mathcal{M}_\ell$. It holds that $M \in \mathcal{M}^2$ if and only if $[M]_\infty$ is integrable, and in the affirmative, $M^2 - [M] \in \mathcal{M}^u$. If M and N are in $\mathcal{M}^2$, then $[M,N]$ is in $\mathcal{V}^i$, in particular the limit $[M,N]_\infty$ exists and is integrable, and $MN - [M,N] \in \mathcal{M}^u$.*

Proof. We begin by proving that $M \in \mathcal{M}^2$ if and only if $[M]_\infty$ is integrable. First assume $M \in \mathcal{M}^2$. We know that $M^2 - [M] \in \mathcal{M}_\ell$. Using Lemma 3.1.6, let (T_n) be a localising sequence with $(M^2 - [M])^{T_n} \in \mathcal{M}^u$. By the optional sampling theorem and Theorem 1.3.1, $E[M]_{T_n} = E[M]^{T_n}_\infty = E(M^{T_n}_\infty)^2 = EM^2_{T_n} \le 4EM^2_\infty$, and then, as $[M]$ is increasing, we obtain $E[M]_\infty = E\lim_n [M]_{T_n} = \lim_n E[M]_{T_n} \le 4EM^2_\infty$ by the monotone convergence theorem, so that $[M]_\infty$ is integrable.

Assume conversely that $[M]_\infty$ is integrable. Let (T_n) be a localising sequence such that $(M^2 - [M])^{T_n} \in \mathcal{M}^u$. Fix $t \ge 0$. We then find that $M^2_{t\wedge T_n} - [M]_{t\wedge T_n}$ is integrable, and by our assumptions, $[M]_{t\wedge T_n}$ is integrable as well. As a consequence, $M^2_{t\wedge T_n}$ is integrable, and it holds that $EM^2_{t\wedge T_n} = E[M]_{t\wedge T_n} \le E[M]_\infty$. Thus, $M^{T_n} \in \mathcal{M}^2$. Applying Theorem 1.3.1, $E\sup_{0\le s\le T_n} M^2_s = E((M^{T_n})^{*2}_\infty) \le 4E(M^{T_n}_\infty)^2 = 4E[M]^{T_n}_\infty = 4E[M]_{T_n}$. Using the monotone convergence theorem, we then obtain $EM^{*2}_\infty \le 4E[M]_\infty$, in particular $\sup_{t\ge 0} EM^2_t$ is finite and so $M \in \mathcal{M}^2$, as desired.

Next, we prove that when $[M]_\infty$ is integrable and $M \in \mathcal{M}^2$, we have $M^2 - [M] \in \mathcal{M}^u$. We use Lemma 1.2.8. First note that $M^2 - [M]$ is has initial value zero and is convergent to an almost sure limit, so the conditions for use of the lemma are satisfied. Let T by any stopping time. As $[M]_\infty$ is integrable and $M \in \mathcal{M}^2$, we know that $M^2_T - [M]_T$ is integrable as well, we need to show that $E(M^2_T - [M]_T)$ is zero. To this end, let (T_n) be a localising sequence such that $(M^2 - [M])^{T_n} \in \mathcal{M}^u$. We then obtain

$$E[M]_T = E\lim_n [M]_{T\wedge T_n} = \lim_n E[M]^{T_n}_T = \lim_n E(M^2_T)^{T_n} = \lim_n EM^2_{T\wedge T_n}.$$

Now, as $(M_T - M_{T\wedge T_n})^2 \le 4M^{*2}_\infty$, which is integrable by Theorem 1.3.1, and $M_{T\wedge T_n}$ converges almost surely to M_T, we find that $M^2_{T\wedge T_n}$ converges in $\mathcal{L}^2$ to M^2_T, so that $EM^2_{T\wedge T_n}$ tends to EM^2_T, allowing us to conclude that $E[M]_T = EM^2_T$ and so Lemma 1.2.8 shows that $M^2 - [M] \in \mathcal{M}^u$.

Finally, consider two elements M and N of $\mathcal{M}^2$. As $[M,N] = \frac{1}{2}([M+N] - [M] - [N])$, we find by our previous results that $[M,N]$ is in $\mathcal{V}^i$ and that the limit $[M,N]_\infty$ exists and is integrable. Noting that $MN - [M,N] = \frac{1}{2}((M+N)^2 - [M+N]) - \frac{1}{2}(M^2 - [M]) - \frac{1}{2}(N^2 - [N])$, we find that that $MN - [M,N]$ is in $\mathcal{M}^u$ as a linear combination of elements in $\mathcal{M}^u$. $\square$

Lemma 3.3.11. *Let $M \in \mathcal{M}^2$. Then $\sum_{0<t}(\Delta M_t)^2$ is integrable.*

Proof. By Theorem 3.3.10, we know that $[M]_\infty$ is integrable. As $[M]$ is increasing and we have $\Delta[M] = (\Delta M)^2$, we obtain $\sum_{0<t}(\Delta M_t)^2 \leq [M]_\infty$, and so $\sum_{0<t}(\Delta M_t)^2$ is integrable as well. □

3.4 Purely discontinuous local martingales

In this final section of the chapter, we use the quadratic covariation in a manner similar to an inner product in order to define the space of purely discontinuous local martingales, which intuitively corresponds to the orthogonal complement of the space of continuous local martingales. We will see that purely discontinuous local martingales corresponds precisely to the subspace of $\mathcal{M}_\ell$ where the quadratic variation can be explicitly computed. Also, we will show that any element of $\mathcal{M}_\ell$ can be uniquely decomposed into a continuous and a purely discontinuous part. This result will prove useful in Chapter 4 when defining the continuous martingale part of a semimartingale.

Definition 3.4.1. *Let $M \in \mathcal{M}_\ell$. We say that M is purely discontinuous if $[M, N]$ is evanescent for all $N \in \mathbf{c}\mathcal{M}_\ell$. The set of purely discontinuous elements of $\mathcal{M}_\ell$ is denoted by $\mathbf{d}\mathcal{M}_\ell$.*

The following two results yield basic properties of purely discontinuous local martingales.

Lemma 3.4.2. *$\mathbf{d}\mathcal{M}_\ell$ is a vector space. If T is a stopping time and $M \in \mathbf{d}\mathcal{M}_\ell$, then $M^T \in \mathbf{d}\mathcal{M}_\ell$ as well.*

Proof. Let $M, N \in \mathbf{d}\mathcal{M}_\ell$ and let $\alpha, \beta \in \mathbb{R}$. Fix $L \in \mathbf{c}\mathcal{M}_\ell$. By Lemma 3.3.8, we have $[\alpha M + \beta N, L] = \alpha[M, L] + \beta[N, L]$, so $[\alpha M + \beta N, L] = 0$ and thus $\alpha M + \beta N$ is in $\mathbf{d}\mathcal{M}_\ell$. We conclude that $\mathbf{d}\mathcal{M}_\ell$ is a vector space. Now let $M \in \mathbf{d}\mathcal{M}_\ell$ and let T be a stopping time. With $L \in \mathbf{c}\mathcal{M}_\ell$, we have $L^T \in \mathbf{c}\mathcal{M}_\ell$ as well, so $[M^T, L] = [M, L^T] = 0$ by Lemma 3.3.8, yielding $M^T \in \mathbf{d}\mathcal{M}_\ell$. □

Lemma 3.4.3. *If M is an element of $\mathcal{M}_\ell$ which is both in $\mathbf{c}\mathcal{M}_\ell$ and in $\mathbf{d}\mathcal{M}_\ell$, then M is evanescent.*

Proof. By definition, we obtain that $[M, M]$ is evanescent, which by Lemma 3.3.8 implies that M is evanescent. □

Next, we show that all elements of $\mathbf{fv}\mathcal{M}_\ell$ are purely discontinuous martingales. This provides a certain level of intuitive understanding of the structure of purely discontinuous local martingales. Afterwards, we prove that any $M \in \mathcal{M}_\ell$ can be decomposed uniquely into a continuous and a purely discontinuous part.

Lemma 3.4.4. *Let $M \in \mathbf{fv}\mathcal{M}_\ell$ and let $N \in \mathcal{M}_\ell$. Almost surely, $\sum_{0<s\le t} \Delta M_s \Delta N_s$ is absolutely convergent for all $t \ge 0$. Furthermore, $[M,N]_t = \sum_{0<s\le t} \Delta M_s \Delta N_s$.*

Proof. That $\sum_{0<s\le t} \Delta M_s \Delta N_s$ is absolutely convergent for all $t \ge 0$ follows from Lemma 3.3.5. To prove the result on the quadratic covariation, first note that for any $M \in \mathbf{fv}\mathcal{M}_\ell$, the integration-by-parts formula applies and yields that $M_t^2 - \sum_{0<s\le t}(\Delta M_s)^2 = 2\int_0^t M_{s-}\,\mathrm{d}M_s$, which is a local martingale by Lemma 3.3.2. Therefore, $[M]_t = \sum_{0<s\le t}(\Delta M_s)^2$ in this case. As a consequence, when $M \in \mathbf{fv}\mathcal{M}_\ell$ and $N \in \mathbf{fv}\mathcal{M}_\ell$, we have

$$\begin{aligned} [M,N]_t &= \frac{1}{4}[M+N]_t - \frac{1}{4}[M-N]_t \\ &= \frac{1}{4}\sum_{0<s\le t}(\Delta M_s + \Delta N_s)^2 - \frac{1}{4}\sum_{0<s\le t}(\Delta M_s - \Delta N_s)^2 = \sum_{0<s\le t}\Delta M_s \Delta N_s. \end{aligned}$$

Next, consider the case where $M \in \mathbf{fv}\mathcal{M}_\ell$ and $N \in \mathcal{M}^b$. By Lemma 3.3.3, $M_t N_t - \int_0^t N_s\,\mathrm{d}M_s$ is in $\mathcal{M}_\ell$. As N is bounded, we obtain $\int_0^t N_s\,\mathrm{d}M_s = \sum_{0<s\le t}\Delta N_s \Delta M_s + \int_0^t N_{s-}\,\mathrm{d}M_s$, where the latter term is in $\mathcal{M}_\ell$ by Lemma 3.3.2. Thus, $MN - \sum_{0<s\le t}\Delta N_s \Delta M_s$ is in $\mathcal{M}_\ell$ and so $[M,N]_t = \sum_{0<s\le t}\Delta M_s \Delta N_s$ in this case.

Considering the case where $N \in \mathcal{M}_\ell^b$, let (T_n) be a localising sequence such that $N^{T_n} \in \mathcal{M}^b$. We then obtain $[M,N]_t^{T_n} = [M,N^{T_n}]_t = \sum_{0<s\le t}\Delta M_s \Delta N_s^{T_n} = \sum_{0<s\le t\wedge T_n}\Delta M_s \Delta N_s$ almost surely, and letting n tend to infinity, we conclude that $[M,N]_t = \sum_{0<s\le t}\Delta M_s \Delta N_s$ in this case as well.

Finally, we consider $M \in \mathbf{fv}\mathcal{M}_\ell$ and $N \in \mathcal{M}_\ell$. By Theorem 3.3.1, there exists $N^b \in \mathcal{M}_\ell^b$ and $N^i \in \mathbf{fv}\mathcal{M}_\ell$ such that $N = N^b + N^i$ almost surely. By what was already shown, we obtain

$$[M,N]_t = [M,N^b]_t + [M,N^i]_t = \sum_{0<s\le t}\Delta M_s \Delta N_s^b + \sum_{0<s\le t}\Delta M_s \Delta N_s^i = \sum_{0<s\le t}\Delta M_s \Delta N_s,$$

up to indistinguishability, as desired. □

Lemma 3.4.5. *It holds that $\mathbf{fv}\mathcal{M}_\ell \subseteq \mathbf{d}\mathcal{M}_\ell$.*

Proof. Let $M \in \mathbf{fv}\mathcal{M}_\ell$ and let $N \in \mathbf{c}\mathcal{M}_\ell$. We then obtain $[M,N]_t = \sum_{0<s\le t}\Delta M_s \Delta N_s = 0$ by Lemma 3.4.4, proving $M \in \mathbf{d}\mathcal{M}_\ell$. □

Lemma 3.4.6. *Let $M \in \mathcal{M}^2$. There exists a purely discontinuous square-integrable martingale M^d with the properties that $\Delta M^d = \Delta M$ and $[M^d]_t = \sum_{0<s\leq t}(\Delta M_s)^2$ almost surely.*

Proof. Let (T_n) be a regular sequence of positive stopping times covering the jumps of M, and define $A^n = \Delta M_{T_n} 1_{(t \geq T_n)}$ and $N^n = A^n - \Pi_p^* A^n$. We will prove the result by showing that $\sum_{k=1}^n N^n$ converges in $\mathcal{M}^2$ to a purely discontinuous square-integrable satisfying the requirements of the lemma.

To this end, first note that $(V_{A^n})_\infty = |\Delta M_{T_n}|$. Therefore, by Lemma 1.3.1, A^n is in $\mathcal{V}$ with $(V_A)_\infty$ square-integrable. Lemma 3.2.7 then shows that $\Pi_p^* A^n$ is in $\mathcal{V}$ with $\Pi_p^* A^n_\infty$ square-integrable, and $E(\Pi_p^* A^n_\infty)^2 \leq 8E(V_{A^n})^2_\infty = 8E(\Delta M_{T_n})^2$. As a consequence, N^n is in $\mathcal{M}^2$ and it holds that $E(N^n_\infty)^2 \leq 2E(A^n_\infty)^2 + 2E(\Pi_p^* A^n_\infty)^2 \leq 16E(\Delta M_{T_n})^2$. Define $M^n = \sum_{i=1}^n N^i$, we wish to argue that (M^n) is a Cauchy sequence in $\mathcal{M}^2$. To obtain this, note that as the graphs of (T_n) are disjoint, we obtain by Lemma 3.4.4 and Lemma 3.2.11 for $k \neq n$ that $[N^n, N^k]_t = \sum_{0<s\leq t} \Delta N^n_s \Delta N^k_s = \sum_{0<s\leq t} \Delta A^n_s \Delta A^k_s = 0$ almost surely. Therefore, by Theorem 3.3.10, $N^n N^k$ is in $\mathcal{M}^u$, in particular $EN^n_\infty N^k_\infty = 0$. For $1 \leq k < n$, we then obtain that

$$\begin{aligned}
\|M^n - M^k\|_2^2 &= E\left(\sum_{i=k+1}^n N^i_\infty\right)^2 = \sum_{i=k+1}^n \sum_{j=k+1}^n EN^i_\infty N^j_\infty \\
&= \sum_{i=k+1}^n E(N^i_\infty)^2 \leq 16 \sum_{i=k+1}^n E(\Delta M_{T_i})^2 \leq 16E\sum_{i=k+1}^\infty (\Delta M_{T_i})^2.
\end{aligned}$$

By Lemma 3.3.11, $\sum_{0<t}(\Delta M_t)^2$ is integrable. The dominated convergence theorem then allows us to conclude that the above tends to zero as k and n tend to infinity. As a consequence, (M^n) is a Cauchy sequence in $\mathcal{M}^2$. Therefore, by Theorem 1.3.4, it converges in $\mathcal{M}^2$ to some limit M^d.

It remains to prove the properties claimed for M^d. We first show that M^d is in $\mathbf{d}\mathcal{M}_\ell$. Consider some $N \in \mathbf{c}\mathcal{M}^b$. We then have in particular that $N \in \mathcal{M}^2$. We wish to argue that $[M^d, N]$ is evanescent. To this end, as $[M^d, N]$ is continuous and has paths of finite variation, it suffices by Theorem 3.1.9 to show that $[M^d, N] \in \mathcal{M}^u$. By Theorem 3.3.10, it holds that $[M^d, N]$ is almost surely convergent. Therefore, to show $[M^d, N] \in \mathcal{M}^u$, it suffices by Lemma 1.2.8 to argue that $E[M^d, N]_T = 0$ for all stopping times T. As $N \in \mathbf{c}\mathcal{M}^b$ here is arbitrary, it suffices to show that $E[M^d, N]_\infty = 0$ for all $N \in \mathbf{c}\mathcal{M}^b$.

To prove $E[M^d, N]_\infty = 0$, first note that $[M^d, N] = [M^d - M^n, N] + [M^n, N] = [M^d - M^n, N]$, by Lemma 3.4.4. Thus, it suffices to prove that $\lim_n E[M^d - M^n, N]_\infty = 0$. By Theorem

3.3.10, we have $E[M^d - M^n, N]_\infty = E(M^d_\infty - M^n_\infty)N_\infty$. As $M^d - M^n$ converges to zero in $\mathcal{M}^2$, $E(M^d - M^n)^2_\infty$ converges to zero by Theorem 1.3.1. Therefore, by the Cauchy-Schwartz inequality, $E(M^d_\infty - M^n_\infty)N_\infty$ also converges to zero. As a consequence, we obtain $E[M^d, N]_\infty = \lim_n E[M^d - M^n, N]_\infty = 0$. Collecting our results, this yields that $[M^d, N]$ is evanescent for all $N \in \mathbf{c}\mathcal{M}^b$. Now consider instead a general element $N \in \mathbf{c}\mathcal{M}_\ell$. By Lemma 3.1.6, there is a localising sequence (T_n) such that $M^{T_n} \in \mathbf{c}\mathcal{M}^b$. We then obtain that $[M^d, N]^{T_n}$ is evanescent by what already was shown, and letting n tend to infinity, we conclude that $[M^d, N]$ is evanescent for any $N \in \mathbf{c}\mathcal{M}_\ell$, proving that $M^d \in \mathbf{d}\mathcal{M}_\ell$, as desired.

Next, we show that $\Delta M^d = \Delta M$. To this end, note that by Theorem 1.3.3, there is a subsequence such that $\sup_{t\geq 0} |M^d_t - M^{n_k}_t|$ converges almost surely to zero. By Lemma A.2.6, $\sup_{t\geq 0} |\Delta M^d_t - \Delta M^{n_k}_t|$ then also converges almost surely to zero. Note that by Lemma 3.2.11, $\Delta M^n = \sum_{i=1}^n \Delta M_{T_i} 1_{[\![T_i]\!]}$ almost surely. Therefore, it also almost surely holds that ΔM^{n_k} converges pointwise to $\sum_{i=1}^\infty \Delta M_{T_i} 1_{[\![T_i]\!]}$. As a consequence, $\Delta M^d = \sum_{i=1}^\infty \Delta M_{T_i} 1_{[\![T_i]\!]} = \Delta M$ almost surely, as desired.

It remains to show that $[M^d]_t = \sum_{0<s\leq t} (\Delta M_s)^2$ almost surely. To this end, note that applying Lemma 3.4.4 twice, we have $[M^d - M^n]_t = [M^d]_t - 2\sum_{0<s\leq t} \Delta M^d_s \Delta M^n_s + \sum_{0<s\leq t}(\Delta M^n_s)^2$. Recalling that $\Delta M^n = \sum_{i=1}^n \Delta M_{T_i} 1_{[\![T_i]\!]}$, we find $\sum_{0<s\leq t} \Delta M^d_s \Delta M^n_s = \sum_{i=1}^n (\Delta M_{T_i})^2 1_{(T_i \leq t)}$ and $\sum_{0<s\leq t}(\Delta M^n_s)^2 = \sum_{i=1}^n (\Delta M_{T_i})^2 1_{(T_i\leq t)}$, so that

$$[M^d - M^n]_t = [M^d]_t - \sum_{i=1}^n (\Delta M_{T_i})^2 1_{(T_i \leq t)}.$$

Now, as $M \in \mathcal{M}^2$, Lemma 3.3.11 shows that $\sum_{0<s\leq t}(\Delta M_s)^2$ is integrable, in particular almost surely finite. Therefore, the above yields $\lim_n [M^d - M^n]_t = [M^d]_t - \sum_{0<s\leq t}(\Delta M_s)^2$, where the limit is almost sure. Furthermore, we obtain that $[M^d - M^n]_t$ is nonnegative and bounded from above by $[M^d]_t$. As $[M^d]_t$ is integrable by Theorem 3.3.10, we may apply the dominated convergence theorem to obtain $E \lim_n [M^d - M^n]_t = \lim_n E[M^d - M^n]_t$. However, as $M^d - M^n$ converges to zero in $\mathcal{M}^2$, $E[M^d - M^n]_\infty$ converges to zero as well. All in all, we conclude $E([M^d]_t - \sum_{0<s\leq t}(\Delta M_s)^2) = E \lim_n [M^d - M^n]_t = 0$, and as the integrand is nonnegative by our earlier observations, this implies $[M^d]_t = \sum_{0<s\leq t}(\Delta M_s)^2$ almost surely, as desired. $\square$

Theorem 3.4.7. *Let $M \in \mathcal{M}_\ell$. There exists processes $M^c \in \mathbf{c}\mathcal{M}_\ell$ and $M^d \in \mathbf{d}\mathcal{M}_\ell$, unique up to indistinguishability, such that $M = M^c + M^d$.*

Proof. Uniqueness follows from Lemma 3.4.3. We prove existence.

First consider the case where $M \in \mathcal{M}^b$. By Lemma 3.4.6, there is $M^d \in \mathbf{d}\mathcal{M}_\ell$ such that $\Delta M^d = \Delta M$ almost surely. Putting $N = M - M^d$, we find that $N \in \mathcal{M}_\ell$ and N is almost surely continuous. Letting F be the null set where N is not continuous, put $M^c = 1_{F^c} N$. Then, $M^c \in \mathbf{c}\mathcal{M}_\ell$ and $M^d \in \mathbf{d}\mathcal{M}_\ell$, and $M = M^c + M^d$ almost surely. This proves existence in the case $M \in \mathcal{M}^b$.

Next, consider the case $M \in \mathcal{M}^b_\ell$. Let (T_n) be a localising sequence such that $M^{T_n} \in \mathcal{M}^b$. From what we already have shown, there exists processes $(M^{T_n})^c \in \mathbf{c}\mathcal{M}_\ell$ and $(M^{T_n})^d \in \mathbf{d}\mathcal{M}_\ell$ such that $M^{T_n} = (M^{T_n})^c + (M^{T_n})^d$ almost surely. As both $\mathbf{c}\mathcal{M}_\ell$ and $\mathbf{d}\mathcal{M}_\ell$ are stable under stopping, uniqueness yields $((M^{T_{n+1}})^c)^{T_n} = (M^{T_n})^c$ and $((M^{T_{n+1}})^d)^{T_n} = (M^{T_n})^d$. Therefore, the processes may be pasted together to processes M^c and M^d in $\mathbf{c}\mathcal{M}_\ell$ and $\mathbf{d}\mathcal{M}_\ell$, respectively, such that $M = M^c + M^d$ almost surely, proving existence in the case $M \in \mathcal{M}^b_\ell$.

Finally, consider a general $M \in \mathcal{M}_\ell$. By Theorem 3.3.1, there exists processes $M^b \in \mathcal{M}^b_\ell$ and $M^v \in \mathbf{fv}\mathcal{M}_\ell$ such that $M = M^b + M^v$ almost surely. By Lemma 3.4.4, M^v is in $\mathbf{d}\mathcal{M}_\ell$. By what was already shown, there exists a decomposition $M^b = (M^b)^c + (M^b)^d$ where $(M^b)^c \in \mathbf{c}\mathcal{M}_\ell$ and $(M^b)^d \in \mathbf{d}\mathcal{M}_\ell$. Therefore, putting $M^c = (M^b)^c$ and $M^d = (M^b)^d + M^v$, we obtain the desired result. □

Theorem 3.4.7 allows us to prove several interesting results both about $\mathcal{M}_\ell$ in general and about $\mathbf{d}\mathcal{M}_\ell$ in particular. Theorem 3.4.8 gives a characterization of $\mathbf{d}\mathcal{M}_\ell$ in terms of the quadratic covariation, while Theorem 3.4.9 shows how the quadratic covariation can be decomposed into two components where one is continuous and the other is the sum of its jumps. Theorem 3.4.11 yields a sufficient criterion for an element of $\mathbf{d}\mathcal{M}_\ell$ to be in $\mathbf{fv}\mathcal{M}_\ell$.

Theorem 3.4.8. *Let $M \in \mathcal{M}_\ell$. The following are equivalent:*

(1). $M \in \mathbf{d}\mathcal{M}_\ell$.

(2). $[M]_t = \sum_{0<s\le t}(\Delta M_s)^2$.

(3). For any $N \in \mathcal{M}_\ell$, $[M,N]_t = \sum_{0<s\le t} \Delta M_s \Delta N_s$.

Proof. **Proof that** (1) **implies** (2)**.** First consider the case where M is purely discontinuous with $M = M^b + M^i$, where $M^b \in \mathcal{M}^b$ and $M^i \in \mathbf{fv}\mathcal{M}_\ell$. We wish to show $[M]_t = \sum_{0<s\le t}(\Delta M_s)^2$. By Lemma 3.4.5, M^i is purely discontinuous. As $M^b = M - M^i$, M^b is purely discontinuous as well. Applying Lemma 3.4.6, we obtain $N \in \mathbf{d}\mathcal{M}_\ell$ with the properties that $\Delta N = \Delta M^b$ and $[N]_t = \sum_{0<s\le t}(\Delta M^b_s)^2$. As $M^b - N$ is continuous while

both M^b and N are purely discontinuous, we conclude that $M^b = N$ almost surely by Lemma 3.4.3, in particular $[M^b]_t = \sum_{0<s\leq t}(\Delta M^b_s)^2$. Applying Lemma 3.4.4, we then obtain

$$\begin{aligned}[M]_t &= [M^b]_t + 2[M^b, M^i]_t + [M^i]_t \\ &= \sum_{0<s\leq t}(\Delta M^b_s)^2 + 2\sum_{0<s\leq t}\Delta M^b_s \Delta M^i_s + \sum_{0<s\leq t}(\Delta M^i_s)^2 = \sum_{0<s\leq t}(\Delta M_s)^2,\end{aligned}$$

proving the result in this case. Now consider an arbitrary $M \in \mathbf{d}\mathcal{M}_\ell$. By Theorem 3.3.1, $M = M^b + M^v$ almost surely, where $M^b \in \mathcal{M}^b_\ell$ and $M^v \in \mathbf{fv}\mathcal{M}_\ell$. Letting T_n be a common localising sequence for M^b and M^v, our previous result implies

$$[M]^{T_n}_t = [M^{T_n}]_t = \sum_{0<s\leq t}(\Delta M^{T_n}_s)^2 = \sum_{0<s\leq t\wedge T_n}(\Delta M_s)^2,$$

so letting n tend to infinity yields the result in the general case.

Proof that (2) implies (3). Now consider $M \in \mathcal{M}_\ell$ such that $[M]_t = \sum_{0<s\leq t}(\Delta M_s)^2$. We wish to argue that for any $N \in \mathcal{M}_\ell$, $[M,N]_t = \sum_{0<s\leq t}\Delta M_s \Delta N_s$. We first show that $M \in \mathbf{d}\mathcal{M}_\ell$. Using Theorem 3.4.7, let $M = M^c + M^d$ be the decomposition of M into its continuous and purely discontinuous parts. By the implication already proven, it holds that $[M^d]_t = \sum_{0<s\leq t}(\Delta M^d_s)^2 = \sum_{0<s\leq t}(\Delta M_s)^2 = [M]_t$. As $[M] = [M^c + M^d] = [M^c] + [M^d]$, we conclude that $[M^c]$ is evanescent. By Lemma 3.3.8, M^c is evanescent. Therefore, $M = M^d$ almost surely, so $M \in \mathbf{d}\mathcal{M}_\ell$.

Now take $N \in \mathcal{M}_\ell$. Using Theorem 3.4.7, let $N = N^c + N^d$ be the decomposition of N into its continuous and purely discontinuous parts. As $M \in \mathbf{d}\mathcal{M}_\ell$, both $M + N^d$ and $M - N^d$ are in $\mathbf{d}\mathcal{M}_\ell$. By the implication already proven, we then obtain

$$\begin{aligned}[M,N]_t &= [M, N^d]_t = \tfrac{1}{4}([M+N^d]_t - [M-N^d]_t) \\ &= \frac{1}{4}\sum_{0<s\leq t}(\Delta M_s + \Delta N^d_s)^2 - (\Delta M_s - \Delta N^d_s)^2 \\ &= \sum_{0<s\leq t}\Delta M_s \Delta N^d_s = \sum_{0<s\leq t}\Delta M_s \Delta N_s,\end{aligned}$$

as desired.

Proof that (3) implies (1). Fix $N \in \mathbf{c}\mathcal{M}_\ell$. As $[M,N] = \sum_{0<s\leq t}\Delta M_s \Delta N_s = 0$ by our assumptions, it follows that $M \in \mathbf{d}\mathcal{M}_\ell$. □

Theorem 3.4.9. *Let $M, N \in \mathcal{M}_\ell$. It holds that $[M,N]_t = [M^c, N^c]_t + \sum_{0<s\leq t}\Delta M_s \Delta N_s$.*

Proof. Fix $M, N \in \mathcal{M}_\ell$. Applying Theorem 3.4.8, we obtain

$$\begin{aligned}[M,N]_t &= [M^c,N^c]_t + [M^c,N^d]_t + [M^d,N^c]_t + [M^d,N^d]_t \\ &= [M^c,N^c]_t + \sum_{0<s\le t} \Delta M^d_s \Delta N^d_s = [M^c,N^c]_t + \sum_{0<s\le t} \Delta M_s \Delta N_s,\end{aligned}$$

as was to be proven. □

Lemma 3.4.10. *Let $M \in \mathcal{M}_\ell$. If $\Delta M \in \mathcal{V}$, then $\Delta M \in \mathcal{V}^i_\ell$.*

Proof. Using Lemma 3.1.5, let (R_n) be a localising sequence such that $(M^b)^{R_n} \in \mathcal{M}^u$, let $S_n = \inf\{t \ge 0 \mid (V_{\Delta M})_t > n\}$ and let $U_n = \inf\{t \ge 0 \mid |M_t| > n\}$. Putting $T_n = R_n \wedge S_n \wedge U_n$, we then obtain

$$(V^{T_n}_{\Delta M})_\infty \le (V_{\Delta M})_{T_n-} + \Delta(V_{\Delta M})_{T_n} \le n + |\Delta M_{T_n}| \le 2n + |M_{T_n}|,$$

which is integrable, so $V^{T_n}_{\Delta M} \in \mathcal{A}^i$ and thus $\Delta M \in \mathcal{V}^i_\ell$. □

Theorem 3.4.11. *Assume that $M \in \mathbf{d}\mathcal{M}_\ell$. If ΔM has finite variation, then M almost surely has paths of finite variation.*

Proof. We know that $M = M^b + M^v$ almost surely, where $M^b \in \mathcal{M}^b_\ell$ and $M^v \in \mathbf{fv}\mathcal{M}_\ell$. It will suffice to show that M^b almost surely has paths of finite variation. To this end, first note that $\Delta M^b = \Delta M - \Delta M^v$. As ΔM and ΔM^v are in $\mathcal{V}$, so is ΔM^b, and so Lemma 3.4.10 shows that $\Delta M^b \in \mathcal{V}^i_\ell$.

As $\Delta M^b \in \mathcal{V}$, we may define $A = \sum_{0<s\le t} \Delta M^b_s$, where the sum converges absolutely for all $t \ge 0$. As $V_A \le 2V_{\Delta M^b}$, we find that $A \in \mathcal{V}^i_\ell$, and so the compensator $\Pi^*_p A$ is well-defined. Put $N = A - \Pi^*_p A$, we then have $N \in \mathbf{fv}\mathcal{M}_\ell$. We claim that N has the same jumps as M. To show this, let (T_n) be a localising sequence such that $A^{T_n} \in \mathcal{V}^i$, $M_{T_n} \in \mathcal{M}^u$ and $N^{T_n} \in \mathcal{M}^u$. Let T be some stopping time. If T is totally inaccessible, $\Delta(\Pi^*_p A)^{T_n}_T = (\Pi^*_p A^{T_n})_T = 0$ almost surely by Theorem 2.3.9, yielding $\Delta N^{T_n}_t = \Delta A^{T_n}_T = \Delta M^{T_n}_T$. If T is predictable, we have $E(\Delta N^{T_n} | \mathcal{F}_{T-}) = 0$ by Lemma 3.1.8 while $(\Pi^*_p A)^{T_n}_T$ is $\mathcal{F}_{T-}$ measurable by Theorem 2.3.9, so

$$\Delta \Pi^*_p A^{T_n}_T = E(\Delta \Pi^*_p A^{T_n}_T | \mathcal{F}_{T-}) = E(\Delta A^{T_n}_T | \mathcal{F}_{T-}) - E(\Delta N^{T_n} | \mathcal{F}_{T-}) = E(\Delta M^{T_n}_T | \mathcal{F}_{T-}) = 0$$

almost surely. Thus, for any stopping time T which is either predictable or totally inaccessible, $\Delta \Pi^*_p A^{T_n}_T$ is almosts surely zero. Applying Theorem 2.3.8, this shows that $\Pi^*_p A^{T_n}$ is almost surely continuous, so $\Pi^*_p A$ is almost surely continuous. We conclude that M^b and N almost surely have the same jumps. As both M^b and N are in $\mathbf{d}\mathcal{M}_\ell$, Lemma 3.4.3 shows that they are indistinguishable. As $N \in \mathbf{fv}\mathcal{M}_\ell$, this proves that M^b almost surely has paths of finite variation. This concludes the proof. □

3.5 Exercises

Exercise 3.5.1. *Assume that X is a continuous adapted process with initial value zero and that S and T are stopping times. Show that if X^T and X^S are in $\mathcal{M}^u$, then $X^{S\wedge T}$ and $X^{S\vee T}$ are in $\mathcal{M}^u$ as well.*

Exercise 3.5.2. *Let $M \in \mathcal{M}_\ell$. Show that $M \in \mathcal{M}^u$ if and only if $(M_T)_{T\in\mathcal{C}}$ is uniformly integrable, where $\mathcal{C} = \{T|T$ is a bounded stopping time$\}$.*

Exercise 3.5.3. *Let M be a local martingale and assume that M_0 is integrable. Show that if $M \geq 0$, then M is a supermartingale.*

Exercise 3.5.4. *Let $M \in \mathcal{M}_\ell$ and define $M^*_t = \sup_{s\leq t}|M_s|$. Show that $M^* \in \mathcal{A}^i_\ell$.*

Exercise 3.5.5. *Let $M \in \mathcal{M}_\ell$. Show that if $\Delta M \geq 0$, then ΔM_T is almost surely zero for all predictable stopping times.*

Exercise 3.5.6. *Let N be an $\mathcal{F}_t$ Poisson process, and let T_n be the n'th jump time of N. Show that T_n is totally inaccessible.*

Exercise 3.5.7. *Let N be an $\mathcal{F}_t$ Poisson process. Show that $N \in \mathcal{A}^i_\ell$ and that $\Pi^*_p N_t = t$ almost surely.*

Exercise 3.5.8. *Let $A \in \mathcal{A}^i_\ell$ and assume that $\Pi^*_p A$ is almost surely continuous. Show that ΔA_T is almost surely zero for all predictable stopping times T.*

Exercise 3.5.9. *Let $A \in \mathcal{V}$. Show that if A is predictable, then $A \in \mathcal{V}^i_\ell$.*

Exercise 3.5.10. *Let N be an $\mathcal{F}_t$ Poisson process and let $M_t = N_t - t$. Show that the process $\int_0^t N_{s-}\,\mathrm{d}M_s$ is in $\mathcal{M}_\ell$ while the process $\int_0^t N_s\,\mathrm{d}M_s$ is not in $\mathcal{M}_\ell$.*

Exercise 3.5.11. *Let T be a totally inaccessible stopping time, and let $A_t = 1_{(t\geq T)}$. Show that $E\exp(-\lambda\Pi^*_p A_T) = 1/(1+\lambda)$ for all $\lambda > 0$.*

Exercise 3.5.12. *Let $M \in \mathcal{M}_\ell$ and let $S \leq T$ be two stopping times. Show that if the equality $[M]_S = [M]_T$ holds almost surely, then $M^T = M^S$ almost surely.*

Exercise 3.5.13. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Let $t \geq 0$ and define $t^n_k = kt2^{-n}$ for $k \leq 2^n$. Show that $\sum_{k=1}^{2^n}(W_{t^n_k} - W_{t^n_{k-1}})^2$ converges in probability to t. Use this to conclude that the convergences $\sum_{k=1}^{2^n} W_{t^n_{k-1}}(W_{t^n_k} - W_{t^n_{k-1}}) \xrightarrow{P} \frac{1}{2}W_t^2 - \frac{1}{2}t$ and $\sum_{k=1}^{2^n} W_{t^n_k}(W_{t^n_k} - W_{t^n_{k-1}}) \xrightarrow{P} \frac{1}{2}W_t^2 + \frac{1}{2}t$ hold as n tends to infinity.*

Exercise 3.5.14. *Define $\mathcal{M}^2_\ell$ as the set of $M \in \mathcal{M}_\ell$ such that there exists a localising sequence (T_n) with $M^{T_n} \in \mathcal{M}^2$. Show that $M \in \mathcal{M}^2_\ell$ if and only if $[M] \in \mathcal{A}^i_\ell$.*

Exercise 3.5.15. *Let* $M, N \in \mathcal{M}^2_\ell$*. Show that* $[M, N] \in \mathcal{V}^i_\ell$*.*

Exercise 3.5.16. *Let* $M, N \in \mathcal{M}^2_\ell$ *and define the predictable quadratic covariation* $\langle M, N \rangle$ *as the compensator of* $[M, N]$*, and define the predictable quadratic variation* $\langle M \rangle$ *as the compensator of* $[M]$*. Show that* M *is evanescent if and only if* $\langle M \rangle$ *is evanescent.*

Exercise 3.5.17. *Let* N *be an* $\mathcal{F}_t$ *Poisson process and let* $M_t = N_t - t$*. Prove that* $[M] = N$*.*

Exercise 3.5.18. *Let* $\mathbf{c}\mathcal{M}^2 = \mathbf{c}\mathcal{M}_\ell \cap \mathcal{M}^2$ *and* $\mathbf{d}\mathcal{M}^2 = \mathbf{d}\mathcal{M}_\ell \cap \mathcal{M}^2$*. Show that for any* $M \in \mathcal{M}^2$*, there exists* $M^c \in \mathbf{c}\mathcal{M}^2$ *and* $M^d \in \mathbf{d}\mathcal{M}^2$ *such that* $M = M^c + M^d$ *almost surely.*

Chapter 4

Stochastic integration

In this chapter, we introduce the space of semimartingales, which will provide us with a natural space of integrators for the stochastic integral, and we define the stochastic integral of a locally bounded predictable process with respect to a semimartingale and consider the basic properties of the integral, in particular proving Itô's formula.

The structure of the chapter is as follows. In Section 4.1 we define the space of semimartingales, we introduce the quadratic variation for semimartingales and we prove some elementary properties. In particular, we introduce the concept of pre-stopping, which is particularly applicable to semimartingales.

In Section 4.2, we define the stochastic integral. The main difficulty is defining the integral with respect to local martingales. Here, the theory developed in Chapter 3 will prove essential. We also prove some elementary properties of the stochastic integral.

Finally, in Section 4.3, we consider some more advanced properties of the stochastic integral, proving the dominated convergence theorem, the integration-by-parts formula as well as Itô's formula.

4.1 Semimartingales

In this section, we define the space of semimartingales and investigate its basic properties.

Definition 4.1.1. *We say that a process X is a semimartingale if it is càdlàg and adapted and there exists $M \in \mathcal{M}_\ell$ and $A \in \mathcal{V}$ such that $X = X_0 + M + A$. The space of semimartingales is denoted by $\mathcal{S}$.*

The following lemmas yield some fundamental results about semimartingales. Lemma 4.1.2 shows that having an almost sure decomposition of a càdlàg process X is sufficient to ensure the semimartingale property, while Lemma 4.1.3 shows that $\mathcal{S}$ is a vector space stable under stopping. Lemma 4.1.4 concerns the level of uniqueness in the decomposition of a semimartingale into its local martingale and finite variation parts, and Lemma 4.1.5 proves the existence of a decomposition with extra regularity properties.

Lemma 4.1.2. *Let X be a càdlàg process such that $X = X_0 + M + A$ almost surely, where $M \in \mathcal{M}_\ell$ and $A \in \mathcal{V}$. Then X is a semimartingale.*

Proof. Let $N = X - X_0 - M - A$, the process N is then an evanescent càdlàg process and therefore an element of $\mathcal{M}_\ell$. We then obtain $X = X_0 + (M + N) + A$, so X is a semimartingale. □

Lemma 4.1.3. *It holds that $\mathcal{S}$ is a vector space. If T is any stopping time and $X \in \mathcal{S}$, then $X^T \in \mathcal{S}$ as well. If $F \in \mathcal{F}_0$ and $X \in \mathcal{S}$, then $1_F X \in \mathcal{S}$ as well.*

Proof. $\mathcal{S}$ is a vector space since $\mathcal{M}_\ell$ and $\mathcal{V}$ are vector spaces. Next, let T be a stopping time and assume that $X \in \mathcal{S}$ with $X = X_0 + M + A$. Then $X^T = X_0 + M^T + A^T$. As $M^T \in \mathcal{M}_\ell$ and $A^T \in \mathcal{V}$, it follows that $X^T \in \mathcal{S}$. Finally, for $F \in \mathcal{F}_0$, we obtain that $1_F X = 1_F X_0 + 1_F M + 1_F A$, where $1_F X_0$ is $\mathcal{F}_0$ measurable, $1_F M \in \mathcal{M}_\ell$ and $1_F A \in \mathcal{V}$. Thus, $1_F X \in \mathcal{S}$. □

Lemma 4.1.4. *Let X be a semimartingale. If $X = X_0 + M + A$ and $X = X_0 + N + B$ are two decompositions of X, it holds that $M - N$ and $A - B$ are in $\mathbf{fv}\mathcal{M}_\ell$.*

Proof. Clearly, $M - N = B - A$. The left-hand side is a local martingale, and the right-hand side is of finite variation. By Theorem 3.1.9, both processes are in $\mathbf{fv}\mathcal{M}_\ell$. □

Lemma 4.1.5. *Let X be a semimartingale. Then, there exists $M \in \mathcal{M}_\ell^b$ and $A \in \mathcal{V}$ such that $X = X_0 + M + A$ up to indistinguishability.*

Proof. Let $X = X_0 + M + A$ be some decomposition of X. By Theorem 3.3.1, we there exists $M^b \in \mathcal{M}_\ell^b$ and $M^v \in \mathbf{fv}\mathcal{M}_\ell$ such that $M = M^b + M^v$ up to indistinguishability. In particular, M^v has paths of finite variation. Thus, we obtain $X = X_0 + M^b + (A + M^v)$ up to indistinguishability, where $M^b \in \mathcal{M}_\ell^b$ and $A + M^v \in \mathcal{V}$. □

Lemma 4.1.5 provides a useful decomposition of semimartingales, and in particular shows that locally, and semimartingale is the sum of a bounded martingale and an adapted càdlàg process of finite variation. We will now introduce pre-stopping and pre-localisation, and show that semimartingales are stable under pre-stopping and pre-locally possesses some very regular features.

Definition 4.1.6. *Let X be any stochastic process, and let T be a stopping time. The process X pre-stopped at T, denoted X^{T-}, is defined by $X^{T-} = X1_{[\![0,T[\![} + X_{T-}1_{[\![T,\infty[\![}$.*

Intuitively, X^{T-} corresponds to stopping X at $T-$, or in other words, just before T, while X^T corresponds to stopping X at T. The connection between the two types of localisation is summarized in the equation $X^{T-} = X^T - \Delta X_T 1_{[\![T,\infty[\![}$. While martingales are stable under stopping, they are in general not stable under pre-stopping. The primary usefulness of pre-stopping is contained in the following three lemmas.

Lemma 4.1.7. *Let X be a semimartingale and let T be a stopping time. Then X^{T-} is a semimartingale as well.*

Proof. Let $X = X_0 + M + A$, where $M \in \mathcal{M}_\ell$ and $A \in \mathcal{V}$. Then $X^{T-} = X^T - \Delta X_T 1_{[\![T,\infty[\![}$, where the latter term is in $\mathcal{V}$. As X^T is a a semimartingale by Lemma 4.1.3, we conclude that X^{T-} is a semimartingale, as desired. □

Lemma 4.1.8. *Let X be any adapted càdlàg process. Define $T_n = \inf\{t \geq 0 || X_t| > n\}$. Then (T_n) is a localising sequence, and $X^{T_n -}1_{(T_n > 0)}$ is bounded by n. In the case where X has initial value zero, T_n is positive and X^{T_n-} is bounded by n.*

Proof. By right-continuity, $|X_{T_n}| \geq n$. As càdlàg mappings are bounded on compacts, this implies that T_n increases to infinity. If $T_n > 0$, we have $|X_t| \leq n$ for $0 \leq t < T_n$. Therefore,

$X^{T_n-}1_{(T_n>0)}$ is bounded by n. In the case where X has initial value zero, it is immediate that T_n is positive. Therefore, $X^{T_n-} = X^{T_n-}1_{(T_n>0)}$, so in this case, X^{T_n-} is also bounded by n. □

We express the content of Lemma 4.1.8 by saying that any adapted càdlàg process is pre-locally bounded.

Lemma 4.1.9. *X be a semimartingale. There exists a localising sequence such that almost surely, X^{T_n-} is the sum of X_0, a bounded martingale and an adapted càdlàg process of bounded variation.*

Proof. It will suffice to consider the case where X has initial value zero. By Lemma 4.1.5, we almost surely have $X = M + A$, where $M \in \mathcal{M}_\ell^b$ and $A \in \mathcal{V}$. Let (T_n) be a localising sequence such that M^{T_n} is bounded. Since V_A is càdlàg adapted, by Lemma 4.1.8 it is pre-locally bounded. Let (S_n) be a localising sequence such that $(V_A)^{S_n-}$ is bounded. Put $U_n = T_n \wedge S_n$, we then have $X^{U_n-} = M^{U_n-} + A^{U_n-} = M^{U_n} + (A^{U_n-} - \Delta M_{U_n} 1_{[\![U_n,\infty[\![})$ almost surely. Here, $M^{U_n} = (M^{T_n})^{S_n}$, so M^{U_n} is a bounded martingale. And since it holds that $\Delta M_{U_n} \leq \sup_{t \leq U_n} \Delta M_t \leq \sup_{t \leq T_n} \Delta M_t$, we find that ΔM_{U_n} is bounded. As a consequence, $\Delta M_{U_n} 1_{[\![U_n,\infty[\![}$ is of bounded variation. And because $(V_A)^{U_n-} \leq (V_A)^{S_n-}$, we find that $(V_A)^{U_n-}$ is bounded, and so A^{U_n-} has bounded variation. We conclude that $A^{U_n-} - \Delta M_{U_n} 1_{[\![U_n,\infty[\![}$ has bounded variation, concluding the proof. □

Our use of pre-stopping will in general be as follows. Consider a localising sequence (T_n). If we are to prove a result solely concerning semimartingales, we first note that if the result holds on $[\![0, T_n[\![$ for all n, it holds for all of $\mathbb{R}_+ \times \Omega$. By Lemma 4.1.7, it will therefore suffice to consider semimartingales pre-stopped at some stopping time. And by Lemma 4.1.9, we may then reduce to the case where $X = X_0 + M + A$, M being a bounded martingale and A being of bounded variation.

Next, we introduce the continuous martingale part of a semimartingale, as well as the quadratic variation and quadratic covariation processes for semimartingales.

Lemma 4.1.10. *For any semimartingale X, there exists a process X^c, called the continuous martingale part of X, unique up to indistinguishability, such that for any decomposition $X = X_0 + M + A$, X^c is equal to M^c, the continuous martingale part of M. We call X^c the continuous martingale part of X.*

Proof. First, we consider uniqueness. Assume that there exists two processes Y and Z with the properties given in the lemma. Let $X = M + A$ be a decomposition of X into its martingale part and finite variation part. Since $Y = M^c$ and $Z = M^c$, where M^c is unique up to evanescence by Theorem 3.4.7, Y and Z are equal up to evanescence.

As regards existence, recall that by Theorem 3.4.7, every local martingale M has a continuous martingale part, unique up to indistinguishability, characterized by $M - M^c$ being a purely discontinuous local martingale. Therefore, the criterion for being the continuous martingale part of a semimartingale is well-defined. It will suffice to consider the case of initial value zero. Let $X = M + A$ be a decomposition of X, and let $X^c = M^c$. We claim that X^c so defined satisfies the requirements of the continuous martingale part of X. To this end, let $X = N + B$ be another decomposition of X. Then $N = M + (N - M)$, where $N - M \in \mathbf{fv}\mathcal{M}_\ell$ by Lemma 4.1.4. In particular, by Lemma 3.4.5, $N - M \in \mathbf{d}\mathcal{M}_\ell$. Therefore, $N^c = M^c$ and X^c is equal to the continuous martingale part of N as well. □

Lemma 4.1.11. *Let X be a semimartingale. Then the sum $\sum_{0<s\leq t}(\Delta X_s)^2$ is absolutely convergent almost surely.*

Proof. By the convention that no process jumps at zero, it will suffice to consider the case with initial value zero. In this case, Let $X = M + A$ be a decomposition of X. From Lemma 3.3.5, we know that the process $\sum_{0<s\leq t}(\Delta M)_s^2$ is almost surely finite. From Lemma A.2.16, we know that the process $\sum_{0<s\leq t}(\Delta A)_s^2$ is almost surely finite. Because we know $(\Delta X)_s^2 = (\Delta M_s + \Delta A_s)^2 \leq 4(\Delta M)_s^2 + 4(\Delta A)_s^2$, the result follows. □

Definition 4.1.12. *Let X be a semimartingale. We define the quadratic variation process of X by $[X]_t = [X^c]_t + \sum_{0<s\leq t}(\Delta X)_s^2$, where $[X^c]$ is the quadratic variation of the local martingale X^c. If Y is another semimartingale, we define the quadratic covariation process of X and Y by $[X,Y]_t = [X^c, Y^c]_t + \sum_{0<s\leq t} \Delta X_s \Delta Y_s$.*

Note that the sum is absolutely convergent by Lemma 4.1.11. Also note that since $[X^c, Y^c]$ is uniquely determined up to evanescence, $[X,Y]$ is uniquely determined up to evanescence as well, and the polarization identity $[X,Y] = \frac{1}{4}([X+Y] - [X-Y])$ holds. Also, it is immediate that $[X] \in \mathcal{A}$ with $\Delta[X] = (\Delta X)^2$ while $[X,Y] \in \mathcal{V}$ with $\Delta[X,Y] = \Delta X \Delta Y$. Further note the similarity of Definition 4.1.12 with the result in Theorem 3.4.9.

Finally, note that for $M, N \in \mathcal{M}_\ell$, their continuous martingale parts as defined in Lemma 4.1.10 coincide with the continuous martingale parts M^c and N^c as given in Theorem 3.4.7. Therefore, by Theorem 3.4.8, the quadratic covariation process for semimartingales M and

N, as given in Definition 4.1.12, is the same as the quadratic covariation process for local martingales of M and N as given in Theorem 3.3.4.

The following lemma is the semimartingale analogoue of Lemma 3.3.8.

Lemma 4.1.13. *Let X and Y be semimartingales, and let T be any stopping time. The quadratic covariation satisfies the following properties up to indistinguishability.*

(1). $[X, X] = [X]$.

(2). $[\cdot, \cdot]$ *is symmetric and linear in both of its coordinates.*

(3). For any $\alpha \in \mathbb{R}$, $[\alpha X] = \alpha^2 [X]$.

(4). It holds that $[X + Y] = [X] + 2[X, Y] + [Y]$.

(5). It holds that $[X, Y]^T = [X^T, Y] = [X, Y^T] = [X^T, Y^T]$.

(6). $[X, Y]$ *is zero if and only if* $X^c Y^c \in \mathcal{M}_\ell$ *and* $\Delta X \Delta Y$ *is evanescent.*

(7). X *has a continuous modification in* $\mathcal{V}$ *if and only if* $[X]$ *is evanescent.*

(8). X *has a continuous modification in* $\mathcal{V}$ *if and only if* $[X, Y]$ *is zero for all* $Y \in \mathcal{S}$.

(9). If $F \in \mathcal{F}_0$, $1_F [X, Y] = [1_F X, Y] = [X, 1_F Y] = [1_F X, 1_F Y]$.

(10). $[X, Y]^{T-} = [X^{T-}, Y] = [X, Y^{T-}] = [X^{T-}, Y^{T-}]$.

Proof. The first five properties are immediate from Lemma 3.3.8 and the definition of the quadratic variation and the quadratic covariation processes. We consider the remaining properties.

Proof of (6). First assume that $[X, Y]$ is zero. Since $[X^c, Y^c]$ is continuous, we obtain

$$\Delta X_t \Delta Y_t = \Delta \left(\sum_{0<s\le t} \Delta X_s \Delta Y_s \right) = \Delta [X, Y]_t = 0,$$

so $\Delta X \Delta Y$ is evanescent. As we then have $[X^c, Y^c] = [X, Y] = 0$ up to evanescence, Lemma 3.3.8 yields that $X^c Y^c$ is a local martingale. Conversely, assume that $X^c Y^c \in \mathcal{M}_\ell$ and that $\Delta X \Delta Y$ is evanescent. We then obtain $[X, Y] = [X^c, Y^c]$, which is evanescent, again by Lemma 3.3.8.

Proof of (7). Assume that $[X]$ is evanescent. By what was previously shown, $(X^c)^2$ is a local martingale and X is almost surely continuous. We wish show that X almost surely has paths of finite variation. To this end, let (T_n) be a localising sequence, we then find that $E(X^c)^2_{t\wedge T_n}=0$ and so $(X^c)^2_{t\wedge T_n}$ is almost surely zero. Therefore, $(X^c)^2_t$ is almost surely zero, and so X^c is evanescent. Next, consider a decomposition $X=X_0+M+A$, where $M\in\mathcal{M}_\ell$ and $A\in\mathcal{V}$. It will suffice to show that M almost surely has paths of finite variation. As X^c is evanescent, $M\in\mathbf{d}\mathcal{M}_\ell$. And as X is almost surely continuous, there is a modification $N\in\mathbf{d}\mathcal{M}_\ell$ of M such that $\Delta N=-\Delta A$. We then obtain $\Delta N\in\mathcal{V}$, so Theorem 3.4.11 shows that N almost surely has paths of finite variation. Therefore, M also almost surely has paths of finite variation, finally yielding that X almost surely has paths of finite variation.

Conversely, if X has a continuous modification in $\mathcal{V}$, we obtain that X^c is evanescent and so $[X]$ is evanescent.

Proof of (8). This follows from the previous result.

Proof of (9). The conclusion is well-defined as $1_F X$ is in $\mathcal{S}$ by Lemma 4.1.3. By the properties already proven for the quadratic covariation, it suffices to demonstrate that almost surely, $1_F[X,Y]=[1_F X,Y]$ for any $F\in\mathcal{F}_0$ and $X,Y\in\mathcal{S}$. To this end, we apply Lemma 3.3.8 to obtain

$$\begin{aligned}
1_F[X,Y]_t &= 1_F[X^c,Y^c]_t+1_F\sum_{0<s\leq t}\Delta X_s\Delta Y_s\\
&= [1_F X^c,Y^c]_t+\sum_{0<s\leq t}1_F\Delta X_s\Delta Y_s=[1_F X,Y]_t
\end{aligned}$$

up to evanescence, as desired.

Proof of (10). By the symmetry and linearity properties already proved, it will suffice to show $[X,Y]^{T-}=[X^{T-},Y]$. To this end, let $X=X_0+M+A$ be a decomposition of X. We then obtain $X^{T-}=X_0+M^{T-}+A^{T-}=X_0+M^T-\Delta M_T 1_{[\![T,\infty[\![}+A^{T-}$. Therefore, the continuous martingale part of X^{T-} is $(X^T)^c$. We then obtain

$$\begin{aligned}
[X,Y]^{T-}_t &= [X,Y]^T_t-1_{(t\geq T)}\Delta X_s\Delta Y_s=[X^T,Y]_t-1_{(t\geq T)}\Delta X_s\Delta Y_s\\
&= [(X^T)^c,Y^c]_t+\sum_{0<s\leq t}\Delta X^T_s\Delta Y_s-1_{(t\geq T)}\Delta X_s\Delta Y_s\\
&= [(X^{T-})^c,Y^c]_t+\sum_{0<s\leq t}\Delta X^{T-}_s\Delta Y_s=[X^{T-},Y]_t,
\end{aligned}$$

proving the result. □

4.2 Construction of the stochastic integral

In this section, we define and prove the existence of the stochastic integral with respect to semimartingales and consider its basic properties. With the tools developed previously, we can almost immediately prove the existence of the stochastic integral of a predictable and locally bounded process with respect to a local martingale.

We begin with a motivating lemma. Consider some $M \in \mathcal{M}_\ell$. Let $S \leq T$ be stopping times and define $H = \xi 1_{]\!]S,T]\!]}$ for some ξ which is bounded and $\mathcal{F}_S$ measurable. The natural definition of the integral of H with respect to M over $[0,t]$ is $\xi(M^T_t - M^S_t)$. Defining L by putting $L = \xi(M^T - M^S)$, we obtain this integral in process form, where L_t intuitively represents the integral of H with respect to M over $[0,t]$. The following lemma shows some properties of this process L.

Lemma 4.2.1. *Let $M \in \mathcal{M}_\ell$, let $S \leq T$ be stopping times, let ξ be bounded and $\mathcal{F}_S$ measurable, and let $H = \xi 1_{]\!]S,T]\!]}$. Then H is predictable. Defining $L = \xi(M^T - M^S)$, it holds that $L \in \mathcal{M}_\ell$, and for all $N \in \mathcal{M}_\ell$, $[L,N] = H \cdot [M,N]$.*

Proof. That H is predictable follows from Lemma 2.2.5. As $S \leq T$, we may also write $L = (\xi(M - M^S))^T$. Since ξ is $\mathcal{F}_S$ measurable, $\xi(M - M^S)$ is in $\mathcal{M}_\ell$ by Lemma 3.3.7, and therefore L is in $\mathcal{M}_\ell$ as well.

It remains to prove $[L,N] = H \cdot [M,N]$ for all $N \in \mathcal{M}_\ell$. To this end, first note that

$$\Delta L_t = \xi(\Delta M^T_t - \Delta M^S_t) = \xi(\Delta M_t 1_{(t\leq T)} - \Delta M_t 1_{(t \leq S)}) = \xi 1_{(S<t\leq T)} \Delta M_t = H_t \Delta M_t,$$

so $\Delta L = H\Delta M$. Thus, $\Delta(H \cdot [M,N]) = H\Delta[M,N] = H\Delta M \Delta N = \Delta L \Delta N$, as required from Theorem 3.3.4. It remains to prove that $LN - H \cdot [M,N]$ is in $\mathcal{M}_\ell$. To do so, we note that by the properties of the ordinary Lebesgue integral, we have

$$\begin{aligned} LN - H \cdot [M,N] &= \xi(M^T - M^S)N - \xi([M,N]^T - [M,N]^S) \\ &= \xi(M^T N - [M,N]^T) - \xi(M^S N - [M,N]^S) \\ &= \xi(M^T N - [M^T,N]) - \xi(M^S N - [M^S,N]), \end{aligned}$$

which is in $\mathcal{M}_\ell$ by Lemma 3.3.7. By the uniqueness statement of Theorem 3.3.4, we may now conclude that $[L,N] = H \cdot [M,N]$ for all $N \in \mathcal{M}_\ell$. □

Lemma 4.2.1 shows that by defining the integral of a simple process of the form $\xi 1_{]\!]S,T]\!]}$ with respect to $M \in \mathcal{M}_\ell$ in a manner corresponding to ordinary integrals, we obtain an element of $\mathcal{M}_\ell$ characterised by a simple quadratic covariation structure in relation to other elements of $\mathcal{M}_\ell$. We take this characterisation as our defining feature of the stochastic integral with respect to integrators in $\mathcal{M}_\ell$. We will later show that this yields an integral which in certain cases may also be interpreted as a particular limit of ordinary Riemann sums.

First, we introduce the space of processes which will function as integrands in the stochastic integral.

Definition 4.2.2. *By $\mathfrak{I}$, we denote the space of predictable processes H such that there is a localising sequence (T_n) with the property that $H^{T_n}1_{(T_n>0)}$ is bounded.*

The reason why we include the indicator $1_{(T_n>0)}$ in Definition 4.2.2 is to ensure that we can integrate sufficiently many processes with nonzero initial value. Note that if we only require that H is predictable with H^{T_n} bounded for some localising sequence for H to be in $\mathfrak{I}$, we would exclude all processes H of the form $H = H_0$ where H_0 is $\mathcal{F}_0$ measurable with a distribution which does not have bounded support. The next two lemmas combine to show that as defined, $\mathfrak{I}$ includes large classes of useful processes, and the lemma following that shows that elements of $\mathfrak{I}$ may be integrated with respect to elements of $\mathcal{V}$.

Lemma 4.2.3. *Assume that X is càdlàg and predictable. Then, there is a localising sequence (T_n) such that $X^{T_n}1_{(T_n>0)}$ is bounded. If X has initial value zero, there is a localising sequence (T_n) such that X^{T_n} is bounded.*

Proof. First, assume that H is càdlàg and predictable. Define $T_n = \inf\{t \geq 0 || \Delta X_t| > n\}$. By Lemma 2.3.7, T_n is a localising sequence of positive, predictable stopping times. For each $n \geq 1$, let (S_n^k) be an announcing sequence for T_n. Since $|\Delta X_t| \leq n$ whenever $0 \leq t < T_n$, we find that $\Delta X^{S_n^k}$ is bounded by n for all k. Define $U_n = \max_{i \leq n} \max_{k \leq n} S_i^k$. Then ΔX^{U_n} is bounded by n. Furthermore, U_n is increasing, and since

$$\sup_n U_n = \sup_n \max_{i \leq n} \max_{k \leq n} S_i^k = \sup_n \sup_k S_n^k = \sup_n T_n,$$

U_n tends to infinity. Thus, U_n is a localising sequence such that X^{U_n} has bounded jumps.

Letting $V_n^k = \inf\{t \geq 0 \mid |X_t^{U_n}| > k\}$, we then obtain that $X^{V_n^k}1_{(V_n^k>0)}$ is bounded. Also, for each n, (V_n^k) tends to infinity. Putting $V_n = \max_{i \leq n} \max_{k \leq n} V_i^k$, we obtain that (V_n) is a localising sequence such that $X^{V_n}1_{(V_n>0)}$ is bounded.

In the case where X has initial value zero, it holds that V_n^k is positive, and so $X^{V_n^k}$ is bounded. Therefore, again defining $V_n = \max_{i \leq n} \max_{k \leq n} V_i^k$, we obtain that (V_n) is a localising sequence such that X^{V_n} is bounded, as desired. $\square$

Lemma 4.2.4. *If H is càglàd and adapted, then $H \in \mathfrak{I}$. If H is càdlàg and predictable, then $H \in \mathfrak{I}$.*

Proof. Consider the case where H is càglàd and adapted. Put $T_n = \inf\{t \geq 0 \mid |H_t| > n\}$. As $(T_n < t) = \cup_{s \in \mathbb{Q}_+, s<t}(|H_s| > n)$, we find that T_n is a stopping time, and as H is pathwisely bounded on compacts, (T_n) is a localising sequence. If $T_n = 0$, $H^{T_n} 1_{(T_n > 0)}$ is zero, and if $T_n > 0$, H^{T_n} is bounded by n. Thus, $H^{T_n} 1_{(T_n > 0)}$ is bounded by n and we conclude $H \in \mathfrak{I}$.

In the case where H is càdlàg and predictable, the desired result follows from Lemma 4.2.3. □

Lemma 4.2.5. *Let $H \in \mathfrak{I}$ and let $A \in \mathcal{V}$. Then H is pathwise Lebesgue integrable with respect to A in the sense of Theorem 1.4.3.*

Proof. Let (T_n) be a localising sequence such that $H^{T_n} 1_{(T_n > 0)}$ is bounded. Letting n tend to infinity, we find that for almost all ω, $H(\omega)$ is bounded on compacts, therefore Lebesgue integrable with respect to $A(\omega)$, as desired. □

We are now ready to carry out the construction of the stochastic integral of an element of $\mathfrak{I}$ with respect to an element of $\mathcal{M}_\ell$. Our objective is to construct a local martingale with a quadratic covariation structure similar to that found in Lemma 4.2.1. The following lemmas will be necessary for the proof and the later proofs of the properties of the stochastic integral.

The following two lemmas consider criteria for identifying the jumps of a local martingale in two particular situations. Note that the integrability conditions in both lemmas precisely match those necessary for the expectations in the proofs to be well-defined.

Lemma 4.2.6. *Let $M \in \mathcal{M}^b$, let $L \in \mathcal{M}^2$ and let H be predictable and bounded. If $E\Delta L_T \Delta N_T = E H_T \Delta M_T \Delta N_T$ for all $N \in \mathcal{M}^2$ and all stopping times T, then $\Delta L = H \Delta M$ almost surely.*

Proof. First note that by Theorem 1.3.1, it holds for any stopping time that ΔL_T and ΔN_T are square-integrable whenever $L, N \in \mathcal{M}^2$. Therefore, both $\Delta L_T \Delta N_T$ and $H_T \Delta M_T \Delta N_T$ are integrable and the criterion in the lemma is well-formed.

To prove the result, note that by Theorem 2.3.8, it suffices to prove that $\Delta L_T = H_T \Delta M_T$ for all positive stopping times which are either predictable or totally inaccessible. And to do so, it suffices by Lemma A.1.21 to show $E\Delta L_T\xi = EH_T\Delta M_T\xi$ for all bounded and $\mathcal{F}_T$ measurable variables ξ.

Consider the case where T is predictable. Take ξ such that ξ is bounded and $\mathcal{F}_T$ measurable. Note that by Lemma 3.1.8, $E\Delta L_T E(\xi|\mathcal{F}_{T-}) = EE(\Delta L_T|\mathcal{F}_{T-})\xi = 0$. Similarly, we also have $EH_T\Delta M_T E(\xi|\mathcal{F}_{T-}) = EH_T E(\Delta M_T|\mathcal{F}_{T-})\xi = 0$, since $H_T 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$ measurable by Lemma 2.2.6. Therefore, to prove $E\Delta L_T\xi = EH_T\Delta M_T\xi$, it suffices to demonstrate that $E\Delta L_T\eta = EH_T\Delta M_T\eta$, where $\eta = \xi - E(\xi|\mathcal{F}_{T-})$. Now, as $E(\eta|\mathcal{F}_{T-}) = 0$, Lemma 3.2.10 and Lemma 3.2.7 yields the existence of $N \in \mathcal{M}^2$ such that $\Delta N_T = \eta$. Therefore, we obtain $E\Delta L_T\eta = E\Delta L_T\Delta N_T = EH_T\Delta M_T\Delta N_T = EH_T\Delta M_T\eta$ and thus $E\Delta L_T\xi = EH_T\Delta M_T\xi$.

Next, consider the case where T is totally inaccesible. Again, Lemma 3.2.10 and Lemma 3.2.7 yields the existence of $N \in \mathcal{M}^2$ such that $\Delta N_T = \xi$. Therefore, we obtain that $E\Delta L_T\xi = E\Delta L_T\Delta N_T = EH_T\Delta M_T\Delta N_T = EH_T\Delta M_T\xi$, as desired.

Theorem 2.3.8 now yields $\Delta L = H\Delta M$ almost surely. □

Lemma 4.2.7. *Let $M \in \mathcal{M}_\ell$, let $L \in \mathcal{M}_\ell$ and let $H \in \mathfrak{I}$. If $\Delta L\Delta N = H\Delta M\Delta N$ almost surely for all $N \in \mathcal{M}_\ell$, then $\Delta L = H\Delta M$ almost surely.*

Proof. We first consider the case where $M \in \mathcal{M}^u$, $L \in \mathcal{M}^u$ and H is bounded and predictable. Using Theorem 2.3.8, we find that to obtain $\Delta L = H\Delta M$, it suffices to show $\Delta L_T = H_T\Delta M_T$ for all positive stopping times which are either predictable or totally inaccessible.

First consider the case where T is totally inacessible. By Lemma 3.2.10, there exists $N \in \mathcal{M}_\ell$ such that $\Delta N_T = 1$, and so $\Delta L_T = H_T\Delta M_T$, as desired.

Next, consider the case where T is predictable. To prove $\Delta L_T = H_T\Delta M_T$, it suffices by Lemma A.1.21 to show $E\Delta L_T\xi = EH_T\Delta M_T\xi$ for all bounded and $\mathcal{F}_T$ measurable variables ξ. Consider such a ξ. Note that by Lemma 3.1.8, $E\Delta L_T E(\xi|\mathcal{F}_{T-}) = EE(\Delta L_T|\mathcal{F}_{T-})\xi = 0$. Similarly, we also have $EH_T\Delta M_T E(\xi|\mathcal{F}_{T-}) = EH_T E(\Delta M_T|\mathcal{F}_{T-})\xi = 0$, since $H_T 1_{(T<\infty)}$ is $\mathcal{F}_{T-}$ measurable by Lemma 2.2.6. Therefore, to prove $E\Delta L_T\xi = EH_T\Delta M_T\xi$, it suffices to prove $E\Delta L_T\eta = EH_T\Delta M_T\eta$ where $\eta = \xi - E(\xi|\mathcal{F}_{T-})$. Now, as $E(\eta|\mathcal{F}_{T-}) = 0$, Lemma 3.2.10 yields the existence of $N \in \mathcal{M}^u$ such that $\Delta N_T = \eta$. As ξ is bounded, η is bounded and so in fact $N \in \mathcal{M}^b$ by the properties given in Lemma 3.2.10. Applying this, we obtain $E\Delta L_T\eta = E\Delta L_T\Delta N_T = EH_T\Delta M_T\Delta N_T = EH_T\Delta M_T\eta$ and thus $E\Delta L_T\xi = EH_T\Delta M_T\xi$.

Summing up, Theorem 2.3.8 now allows us to conclude that $\Delta L = H\Delta M$ almost surely.

This concludes the proof in the case where H is bounded and $L, M \in \mathcal{M}^u$.

Now consider the general case, where $H \in \mathfrak{I}$ and $L, M \in \mathcal{M}^u$. Let (T_n) be a localising sequence such that $H^{T_n} 1_{(T_n>0)}$ is bounded, $L^{T_n} \in \mathcal{M}^u$ and $M^{T_n} \in \mathcal{M}^u$. Then, $L^{T_n} 1_{(T_n>0)}$ and $M^{T_n} 1_{(T_n>0)}$ are in $\mathcal{M}^u$ as well, and for all $N \in \mathcal{M}_\ell$, we have

$$\begin{aligned}\Delta(L^{T_n} 1_{(T_n>0)})\Delta N &= 1_{(T_n>0)}(\Delta L)^{T_n}\Delta N = 1_{(T_n>0)}(\Delta L \Delta N)^{T_n} \\ &= 1_{(T_n>0)}(H\Delta M \Delta N)^{T_n} = 1_{(T_n>0)} H^{T_n}(\Delta M)^{T_n}\Delta N \\ &= (H^{T_n} 1_{(T_n>0)})\Delta(M^{T_n} 1_{(T_n>0)})\Delta N\end{aligned}$$

almost surely. Therefore, $\Delta(L^{T_n} 1_{(T_n>0)}) = H^{T_n} 1_{(T_n>0)}\Delta(M^{T_n} 1_{(T_n>0)})$ almost surely by what we already have shown. Letting n tend to infinity, we obtain that $\Delta L = H\Delta M$ almost surely, as desired. □

Theorem 4.2.8. *Let $M \in \mathcal{M}_\ell$ and let $H \in \mathfrak{I}$. Then, there exists a process $H \cdot M$ in $\mathcal{M}_\ell$, unique up to indistinguishability, such that $[H \cdot M, N] = H \cdot [M, N]$ for all $N \in \mathcal{M}_\ell$. We refer to $H \cdot M$ as the stochastic integral of H with respect to M.*

Proof. We first consider uniqueness. Assume that we have two processes L and K in $\mathcal{M}_\ell$ such that $[L, N] = H \cdot [M, N]$ and $[K, N] = H \cdot [M, N]$ for all $N \in \mathcal{M}_\ell$. In particular, $[L, N] = [K, N]$ for all $N \in \mathcal{M}_\ell$, yielding that $[L - K, N]$ is evanescent for all $N \in \mathcal{M}_\ell$. By Lemma 3.3.8, this implies that L and K are indistinguishable.

Next, we turn to the proof of existence.

Step 1. The case of H bounded and M bounded. Assume that H is bounded and that $M \in \mathcal{M}^b$. In particular, $M \in \mathcal{M}^2$. For any $N \in \mathcal{M}^2$, we may apply Theorem 3.3.9 and the Cauchy-Schwartz inequality to obtain

$$\begin{aligned}E\int_0^\infty |H_s||\,\mathrm{d}[M,N]_s| &\le E\left(\int_0^\infty H_s^2\,\mathrm{d}[M]_s\right)^{\frac{1}{2}}([N]_\infty)^{\frac{1}{2}} \\ &\le \left(E\int_0^\infty H_s^2\,\mathrm{d}[M]_s\right)^{\frac{1}{2}}(E[N]_\infty)^{\frac{1}{2}},\end{aligned}$$

which is finite by our assumptions. Therefore, we may define a function $\varphi : \mathcal{M}^2 \to \mathbb{R}$ by putting $\varphi(N) = E\int_0^\infty H_s\,\mathrm{d}[M,N]_s$. The mapping φ is linear, and as

$$|\varphi(N)| \le E\int_0^\infty |H_s||\,\mathrm{d}[M,N]_s| \le \left(E\int_0^\infty H_s^2\,\mathrm{d}[M]_s\right)^{\frac{1}{2}}\|N\|_2,$$

by what we previously showed, it is also continuous. Therefore, Theorem 1.3.5 shows that there exists $L \in \mathcal{M}^2$ such that for all $N \in \mathcal{M}^2$, $\varphi(N) = EN_\infty L_\infty$. We claim that L satisfies the criteria for being the stochastic integral of H with respect to M. To prove this, we need to show that $[L, N] = H \cdot [M, N]$ for all $N \in \mathcal{M}_\ell$. To do so, we first prove this in the case where $N \in \mathcal{M}^2$.

Fix $N \in \mathcal{M}^2$. As we also have $L \in \mathcal{M}^2$, Theorem 3.3.10 shows that $LN - [L, N] \in \mathcal{M}^u$, and therefore $E[L, N]_\infty = E \int_0^\infty H_s \, \mathrm{d}[M, N]_s$. As this holds for any $N \in \mathcal{M}^2$, we also obtain for all stopping times T that

$$E[L, N]_T = E[L, N^T]_\infty = E \int_0^\infty H_s \, \mathrm{d}[M, N^T]_s = E \int_0^T H_s \, \mathrm{d}[M, N]_s.$$

Therefore, by Lemma 1.2.8, the process $[L, N]_t - \int_0^t H_s \, \mathrm{d}[M, N]_s$ is in $\mathcal{M}^u$. We will show that it is also continuous, this will yield that the process in fact is evanescent. To prove continuity, note that the jump process of this martingale is $\Delta L_t \Delta N_t - H_t \Delta M_t \Delta N_t$. Now fix a stopping time T. As $L, N \in \mathcal{M}^2$, Theorem 1.3.1 shows that ΔL_T and ΔN_T are square-integrable for all stopping times T. Thus, we obtain that $E\Delta L_T \Delta N_T = EH_T \Delta M_T \Delta N_T$ for all stopping times T and all $N \in \mathcal{M}^2$. Lemma 4.2.6 then shows that $\Delta L = H \Delta M$ almost surely. As a consequence, $[L, N]_t - \int_0^t H_s \, \mathrm{d}[M, N]_s$ is in $\mathcal{M}^u$ and almost surely has continuous paths of finite variation. By Theorem 3.1.9, we then find that the process is evanescent. Thus, $[L, N] = H \cdot [M, N]$ for all $N \in \mathcal{M}^2$. In order to prove that L is the stochastic integral of H with respect to M, it remains to extend this to all $N \in \mathcal{M}_\ell$. To this end, first note that if $N \in \mathbf{fv}\mathcal{M}_\ell$, Lemma 3.4.4 yields

$$[L, N]_t = \sum_{0<s\le t} \Delta L_s \Delta N_s = \sum_{0<s\le t} H_s \Delta M_s \Delta N_s = \int_0^t H_s \, \mathrm{d}[M, N]_s.$$

Now take any $N \in \mathcal{M}_\ell$. By Theorem 3.3.1, we have $N = N^b + N^v$ almost surely, where $N^b \in \mathcal{M}_\ell^b$ and $N^v \in \mathbf{fv}\mathcal{M}_\ell$. Let (T_n) be a localising sequence such that $(N^b)^{T_n} \in \mathcal{M}^b$, we then obtain $[L, N^b]^{T_n} = [L, (N^b)^{T_n}] = H \cdot [M, (N^b)^{T_n}] = (H \cdot [M, N^b])^{T_n}$. Letting n tend to infinity, we get $[L, N^b] = H \cdot [M, N^b]$. All in all, we then obtain

$$[L, N] = [L, N^b] + [L, N^v] = H \cdot [M, N^b] + H \cdot [M, N^v] = H \cdot [M, N],$$

as desired. This proves existence in the case where H and M are bounded.

Step 2. The case of H bounded and M locally bounded. We now retain the assumption that H is bounded while considering $M \in \mathcal{M}_\ell^b$. Let (T_n) be a localising sequence such that $M^{T_n} \in \mathcal{M}^b$. By what we already have shown, there exists $L^n \in \mathcal{M}_\ell$ such that for any $N \in \mathcal{M}_\ell$, $[L^n, N] = H \cdot [M^{T_n}, N]$. In particular, we have $[(L^{n+1})^{T_n}, N] = H \cdot [M^{T_n}, N]$, which by uniqueness yields that $(L^{n+1})^{T_n}$ is indistinguishable from L^n. Therefore, may paste the processes (L^n) together to a process $L \in \mathcal{M}_\ell$ such that $L^{T_n} = L^n$ for all $n \geq 1$. Fixing $N \in \mathcal{M}_\ell$, we then obtain $[L,N]^{T_n} = [L^n, N] = H \cdot [M^{T_n}, N] = (H \cdot [M,N])^{T_n}$, and letting n tend to infinity, $[L,N] = H \cdot [M,N]$. This proves existence in this case.

Step 3. The case of H bounded and $M \in \mathcal{M}_\ell$. Now consider the general case, where we merely assume $M \in \mathcal{M}_\ell$, while still assuming that H is bounded. By Theorem 3.3.1, we have $M = M^b + M^v$ almost surely, where $M^b \in \mathcal{M}_\ell^b$ and $M^v \in \mathbf{fv}\mathcal{M}_\ell$. By what was already shown, there exists $L^b \in \mathcal{M}_\ell$ such that $[L^b, N] = H \cdot [M^b, N]$ for all $N \in \mathcal{M}_\ell$. Define $L^v = H \cdot M^v$ as the pathwise Lebesgue integral given in Theorem 1.4.3. By Lemma 3.3.2, $L^v \in \mathcal{M}_\ell$, and by Lemma 3.4.4, we have

$$[L^v, N]_t = \sum_{0<s\leq t} \Delta L^v_s \Delta N_t = \sum_{0<s\leq t} H_s \Delta M^v_s \Delta N_t = \int_0^t H_s \,\mathrm{d}[M^v, N]_s$$

for all $N \in \mathcal{M}_\ell$. Putting $L = L^b + L^v$, we obtain for all $N \in \mathcal{M}_\ell$ that

$$[L,N] = [L^b, N] + [L^v, N] = H \cdot [M^b, N] + H \cdot [M^v, N] = H \cdot [M,N],$$

completing the construction in this case as well.

Step 4. The general case. Now merely assume that $H \in \mathfrak{I}$ and $M \in \mathcal{M}_\ell$. Let (T_n) be a localising sequence such that $H^{T_n} 1_{(T_n>0)}$ is bounded. By what we already have shown, there exists for each n a process $L^n \in \mathcal{M}_\ell$ such that for any $N \in \mathcal{M}_\ell$, $[L^n, N] = H^{T_n} 1_{(T_n>0)} \cdot [M,N]$. By Lemma 3.3.8, This implies

$$\begin{aligned} [(L^{n+1})^{T_n} 1_{(T_n>0)}, N] &= 1_{(T_n>0)} [L^{n+1}, N]^{T_n} = 1_{(T_n>0)} (H^{T_{n+1}} 1_{(T_{n+1}>0)} \cdot [M,N])^{T_n} \\ &= 1_{(T_n>0)} (H^{T_n} 1_{(T_{n+1}>0)} \cdot [M,N]) = H^{T_n} 1_{(T_n>0)} \cdot [M,N]. \end{aligned}$$

By uniqueness, we obtain $(L^{n+1})^{T_n} 1_{(T_n>0)} = L^n$. Therefore, we may paste the processes (L^n) together to a process L such that for all $n \geq 1$, $L^{T_n} 1_{(T_n>0)} = L^n$ almost surely. This in particular shows by Lemma 3.1.7 that $L \in \mathcal{M}_\ell$. As we also have

$$\begin{aligned} [L,N]^{T_n} 1_{(T_n>0)} &= [L^{T_n} 1_{(T_n>0)}, N] = [L^n, N] \\ &= H^{T_n} 1_{(T_n>0)} \cdot [M,N] = (H \cdot [M,N])^{T_n} 1_{(T_n>0)}, \end{aligned}$$

almost surely, we obtain by letting n tend to infinity that $[L,N] = H \cdot [M,N]$. This proves existence in the general case and thus completes the proof. □

Before proceeding to extend the stochastic integral to the case of semimartingale integrators, we prove that the integral as defined coincides with the pathwise Lebesgue integral whenever the integrator is of finite variation.

Theorem 4.2.9. *Let $M \in \mathbf{fv}\mathcal{M}_\ell$ be a local martingale with paths of finite variation and let $H \in \mathfrak{I}$. Then H is integrable with respect to M in the sense of stochastic integration as defined Theorem 4.2.8, and H is pathwise Lebesgue integrable with respect to M in the sense of Theorem 1.4.3, and the two integral processes coincide up to evanescence.*

Proof. By Lemma 4.2.5, H is pathwise Lebesgue integrable with respect to M in the sense of Theorem 1.4.3. We need to prove agreement of the two integral processes. Let X be the stochastic integral of H with respect to M as given in Theorem 4.2.8, and let Y be the pathwise Lebesgue integral of H with respect to M as given in Theorem 1.4.3. We wish to argue that X and Y are indistinguishable.

By construction, we have $[X, N] = H \cdot [M, N]$ for all $N \in \mathcal{M}_\ell$. In particular, for $N \in \mathbf{c}\mathcal{M}_\ell$, $[X, N]$ is evanescent. Therefore, $X \in \mathbf{d}\mathcal{M}_\ell$. Furthermore, by Lemma 3.3.2, $Y \in \mathcal{M}_\ell$, and it is purely discontinuous by Lemma 3.4.5.

Next, fix $N \in \mathcal{M}_\ell$. As $[X, N] = H \cdot [M, N]$ almost surely by construction, we also obtain $\Delta X \Delta N = H \Delta M \Delta N$ almost surely. Therefore, by Lemma 4.2.7, $\Delta X = H \Delta M = \Delta Y$ almost surely. Thus, $X - Y \in \mathbf{d}\mathcal{M}_\ell$ and $X - Y$ is almost surely continuous. By Lemma 3.4.3, X and Y are indistinguishable, as desired. □

With Lemma 4.2.5 and Theorem 4.2.8 at our disposal, the construction of the stochastic integral with respect to semimartingales is a simple task. This existence is the subject of Theorem 4.2.10. After the proof, we will spend the remainder of the section outlining the basic properties of the stochastic integral.

Theorem 4.2.10. *Let $X \in \mathcal{S}$ and let $H \in \mathfrak{I}$. There exists a process $H \cdot X \in \mathcal{S}$, unique up to evanescence, such that for any decomposition $X = X_0 + M + A$ with $M \in \mathcal{M}_\ell$ and $A \in \mathcal{V}$, it holds that H is integrable with respect to M in the sense of Theorem 4.2.8, H is integrable with respect to A in the sense of Theorem 1.4.3, and*

$$(H \cdot X)_t = (H \cdot M)_t + (H \cdot A)_t,$$

We refer to $H \cdot X$ the stochastic integral of H with respect to X.

Proof. Fixing a decomposition $X = X_0 + M + A$, we find that by Theorem 4.2.8, H is integrable with respect to M, and by Lemma 4.2.5, H is integrable with respect to A. To prove the theorem, it will suffice to prove that if $X = X_0 + M + A$ and $X = X_0 + N + B$ are two decompositions with M and N in $\mathcal{M}_\ell$ and A and B in $\mathcal{V}$, then

$$(H \cdot M)_t + (H \cdot A)_t = (H \cdot N)_t + (H \cdot B)_t$$

up to evanescence. To this end, recall from Lemma 4.1.4 that $M-N$ and $A-B$ are in $\mathbf{fv}\mathcal{M}_\ell$, and we have $M-N = B-A$. Applying Theorem 4.2.9 and the linearity of the ordinary Lebesgue integral, we obtain

$$(H \cdot M)_t - (H \cdot N)_t = (H \cdot (M-N))_t = (H \cdot (B-A))_t = (H \cdot B)_t - (H \cdot A)_t,$$

so $(H \cdot M)_t + (H \cdot A)_t = (H \cdot N)_t + (H \cdot B)_t$, as desired. This proves the result. $\square$

Note that we cannot define the stochastic integral with respect to a semimartingale using only the quadratic covariation as we did in Theorem 4.2.8, since the quadratic covariation by Lemma 4.1.13 is invariant with respect to addition of a continuous finite variation process. Further note that the stochastic integral $H \cdot X$ always has initial value zero and does not depend on the initial value of X. The following lemma yields the main properties of the stochastic integral.

Lemma 4.2.11. *Let X and Y be semimartingales, let $H, K \in \mathfrak{I}$ and let T be a stopping time. The stochastic integral with respect to X has the following properties up to evanescence:*

(1). $\mathfrak{I}$ is a linear space, and $H \cdot X$ is a linear mapping in both H and X.

(2). $H \cdot X \in \mathcal{S}$ with decomposition $H \cdot X = H \cdot M + H \cdot A$.

(3). $H 1_{[\![0,T]\!]}$ is in $\mathfrak{I}$ and $(H \cdot X)^T = H 1_{[\![0,T]\!]} \cdot X = H \cdot X^T$.

(4). It holds that $KH \in \mathfrak{I}$ and $K \cdot (H \cdot X) = KH \cdot X$.

(5). For any $H \in \mathfrak{I}$, $\Delta(H \cdot X) = H \Delta X$.

(6). It holds that $(H \cdot X)^c = H \cdot X^c$.

(7). We have $[H \cdot X, Y] = H \cdot [X, Y]$ and $[H \cdot X] = H^2 \cdot [X]$.

(8). If X has finite variation, $H \cdot X$ coincides with the pathwise Lebesgue integral.

(9). If $F \in \mathcal{F}_0$, it holds that $1_F H \in \mathfrak{I}$, $1_F X \in \mathcal{S}$ and $(1_F H) \cdot X = 1_F (H \cdot X) = H \cdot (1_F X)$.

(10). If $H^T = K^T$, then $(H \cdot X)^T = (K \cdot X)^T$.

Proof. **Proof of (1).** Let $\alpha, \beta \in \mathbb{R}$. As $H, K \in \mathfrak{I}$, there are localising sequences (T_n) and (S_n) such that $H^{T_n} 1_{(T_n > 0)}$ and $K^{S_n} 1_{(S_n > 0)}$ are bounded. Using Lemma 3.1.2, $(S_n \wedge T_n)$ is also a localising sequence, and we find that

$$\begin{aligned}(\alpha H + \beta K)^{S_n \wedge T_n} 1_{(S_n \wedge T_n > 0)} &= (\alpha H^{S_n \wedge T_n} + \beta K^{S_n \wedge T_n}) 1_{(S_n > 0)} 1_{(T_n > 0)} \\ &= \alpha (H^{T_n} 1_{(T_n > 0)})^{S_n} 1_{(S_n > 0)} + \beta (K^{S_n} 1_{(S_n > 0)})^{T_n} 1_{(T_n > 0)},\end{aligned}$$

and since $H^{T_n} 1_{(T_n > 0)}$ and $K^{S_n} 1_{(S_n > 0)}$ are bounded, this shows that $\alpha H + \beta K$ is in $\mathfrak{I}$. It remains to prove that $H \cdot X$ is linear in both the integrand H and the integrator X. We first fix X with decomposition $X = X_0 + M + A$ and consider the integral as a mapping in H. We commence by showing that $(\alpha H + \beta K) \cdot M = \alpha(H \cdot M) + \beta(K \cdot M)$. Let $N \in \mathcal{M}_\ell$ be given, we then have, using the characterisation in Theorem 4.2.8 and the properties of ordinary Lebesgue integrals,

$$\begin{aligned}[\alpha(H \cdot M) + \beta(K \cdot M), N] &= \alpha[H \cdot M, N] + \beta[K \cdot M, N] \\ &= \alpha(H \cdot [M, N]) + \beta(K \cdot [M, N]) \\ &= (\alpha H + \beta K) \cdot [M, N] = [(\alpha H + \beta K) \cdot M, N],\end{aligned}$$

so by Lemma 3.3.8, $(\alpha H + \beta K) \cdot M = \alpha(H \cdot M) + \beta(K \cdot M)$, as desired. As we also have $(\alpha H + \beta K) \cdot A = \alpha(H \cdot A) + \beta(H \cdot A)$ when $A \in \mathcal{V}$, this proves that the stochastic integral is linear in the integrand. Next, we prove that it is linear in the integrator. Fix $H \in \mathfrak{I}$, we consider X and Y in $\mathcal{S}$ and wish to prove that $H \cdot (\alpha X + \beta Y) = \alpha(H \cdot X) + \beta(H \cdot Y)$. Assume that we have decompositions $X = X_0 + M + A$ and $Y = Y_0 + N + B$. We first prove that $H \cdot (\alpha M + \beta N) = \alpha(H \cdot M) + \beta(H \cdot N)$. Fixing any $N' \in \mathcal{M}_\ell$, we have

$$\begin{aligned}[\alpha(H \cdot M) + \beta(H \cdot N), N'] &= \alpha[H \cdot M, N'] + \beta[H \cdot N, N'] \\ &= \alpha(H \cdot [M, N']) + \beta(H \cdot [N, N']) \\ &= H \cdot [\alpha M + \beta N, N'] = [H \cdot (\alpha M + \beta N), N'],\end{aligned}$$

so that by Lemma 3.3.8, $H \cdot (\alpha M + \beta N) = \alpha(H \cdot M) + \beta(H \cdot N)$. Therefore, as $\alpha X + \beta Y$ has martingale part $\alpha M + \beta N$ and finite variation part $\alpha A + \beta B$, we obtain, using what was just proven as well as the linearity properties of ordinary Lebesgue integrals, that

$$\begin{aligned}H \cdot (\alpha X + \beta Y) &= H \cdot (\alpha M + \beta N) + H \cdot (\alpha A + \beta B) \\ &= \alpha(H \cdot M) + \beta(H \cdot N) + \alpha(H \cdot A) + \beta(H \cdot B) \\ &= \alpha(H \cdot X) + \beta(H \cdot Y),\end{aligned}$$

as desired.

Proof of (2). This follows immediately from the construction of the integral.

Proof of (3). Assume that $H \in \mathfrak{I}$, we first show that $H1_{[\![0,T]\!]}$ is in $\mathfrak{I}$ as well. By Lemma 2.1.3, $[\![0,T]\!]$ is predictable, and therefore, $H1_{[\![0,T]\!]}$ is predictable. Let (T_n) be a localising sequence such that $H^{T_n}1_{(T_n>0)}$ is bounded. Then $(H1_{[\![0,T]\!]})^{T_n}1_{(T_n>0)} = H^{T_n}1_{(T_n>0)}1_{[\![0,T]\!]}$ is bounded as well. We conclude that $H1_{[\![0,T]\!]} \in \mathfrak{I}$, as desired. In order to prove the identities for the stochastic integral, let $X \in \mathcal{S}$ with decomposition $X = X_0 + M + A$. We then have

$$(H \cdot A)_t^T = \int_0^{T\wedge t} H_s \,\mathrm{d}A_s = \int_0^t (H1_{[\![0,T]\!]})_s \,\mathrm{d}A_s = \int_0^t H_s \,\mathrm{d}A_s^T,$$

so that $(H \cdot A)^T = (H1_{[\![0,T]\!]}) \cdot A = H \cdot A^T$. As regards the martingale part, let $N \in \mathcal{M}_\ell$, then $[(H \cdot M)^T, N] = [H \cdot M, N]^T = (H \cdot [M,N])^T$. Therefore, $[(H \cdot M)^T, N] = H1_{[\![0,T]\!]} \cdot [M,N]$, proving $(H \cdot M)^T = H1_{[\![0,T]\!]} \cdot M$, and $[(H \cdot M)^T, N] = H \cdot [M^T, N]$, which shows that $(H \cdot M)^T = H \cdot M^T$. Collecting our results for the martingale and finite variation parts and using linearity, the result follows.

Proof of (4). As $H, K \in \mathfrak{I}$, we know that there exists a localising sequence (T_n) such that $H^{T_n}1_{(T_n>0)}$ and $K^{T_n}1_{(T_n>0)}$ are bounded. As $(KH)^{T_n}1_{(T_n>0)} = K^{T_n}1_{(T_n>0)}H^{T_n}1_{(T_n>0)}$, and KH is predictable, we conclude $KH \in \mathfrak{I}$. As regards the integral identity, assume that X has decomposition $X = X_0 + M + A$. By the properties of ordinary Lebesgue integrals, $K \cdot (H \cdot A) = KH \cdot A$. As regards the martingale parts, let $N \in \mathcal{M}_\ell$, we then have

$$[K \cdot (H \cdot M), N] = K \cdot [H \cdot M, N] = K \cdot (H \cdot [M,N]) = KH \cdot [M,N],$$

which shows that $K \cdot (H \cdot M)$ satisfies the criterion for being the stochastic integral of KH with respect to M, so $K \cdot (H \cdot M) = KH \cdot M$. Collecting our results and using linearity of the integral in the integrator, we find

$$\begin{aligned} K \cdot (H \cdot X) &= K \cdot (H \cdot M + H \cdot A) = K \cdot (H \cdot M) + K \cdot (H \cdot A) \\ &= KH \cdot M + KH \cdot A = KH \cdot X, \end{aligned}$$

as desired.

Proof of (5). Let $X = X_0 + M + A$. Fixing $N \in \mathcal{M}_\ell$, we have $[H \cdot M, N] = H \cdot [M,N]$ and thus $\Delta(H \cdot M)\Delta N = H\Delta M \Delta N$. Lemma 4.2.7 then shows that $\Delta(H \cdot M) = H\Delta M$. By the properties of the Lebesgue integral, we also have $\Delta(H \cdot A) = H\Delta A$. Thus, we obtain $\Delta(H \cdot X) = \Delta(H \cdot M) + \Delta(H \cdot A) = H\Delta M + H\Delta A = H\Delta X$, as was to be shown.

Proof of (6). Let $X = X_0 + M + A$ and recall that $H \cdot X = H \cdot M + H \cdot A$, where $H \cdot M \in \mathcal{M}_\ell$ and $H \cdot A \in \mathcal{V}$. Let $M = M^c + M^d$ be the decomposition of M into its continuous and purely discontinuous parts. We then also have $H \cdot M = H \cdot M^c + H \cdot M^d$. As $\Delta(H \cdot M^c) = H\Delta M^c = 0$, $H \cdot M^c \in \mathbf{c}\mathcal{M}_\ell$. And as $[H \cdot M^d, N] = H \cdot [M^d, N]$ for all

$N \in \mathcal{M}_\ell$, $H \cdot M^d$ is purely discontinuous. Therefore, $H \cdot M^c$ is the continuous martingale part of $H \cdot X$. As M^c is the continuous martingale part of X, we obtain $(H \cdot X)^c = H \cdot X^c$.

Proof of (7). Let $X = X_0 + M + A$ and $Y = Y_0 + M + A$. By what was already shown,

$$\begin{aligned}[H \cdot X, Y]_t &= [(H \cdot X)^c, Y^c]_t + \sum_{0<s\leq t} \Delta(H \cdot X)_s \Delta Y_s \\ &= [H \cdot X^c, Y^c]_t + \sum_{0<s\leq t} H_s \Delta X_s \Delta Y_s \\ &= H \cdot [X^c, Y^c]_t + \sum_{0<s\leq t} H_s \Delta X_s \Delta Y_s = (H \cdot [X,Y])_t,\end{aligned}$$

proving the first equality. As a consequence, we then also obtain

$$\begin{aligned}[H \cdot X] &= [H \cdot X, H \cdot X] = H \cdot [X, H \cdot X] \\ &= H \cdot [H \cdot X, X] = H \cdot (H \cdot [X,X]) = H^2 \cdot [X].\end{aligned}$$

Proof of (8). This follows from the construction in Theorem 4.2.10 and the result in Theorem 4.2.9.

Proof of (9). First note that $1_F H \in \mathfrak{I}$ as 1_F is predictable, and by Lemma 4.1.3, $1_F X \in \mathcal{S}$. Let $X = X_0 + M + A$. By the properties of ordinary Lebesgue integrals, we know that $(1_F H) \cdot A = 1_F (H \cdot A) = H \cdot (1_F A)$ up to indistinguishability. Therefore, it suffices to prove $(1_F H) \cdot M = 1_F (H \cdot M) = H \cdot (1_F M)$. By Lemma 3.1.3, all three processes are in $\mathcal{M}_\ell$. Therefore, it suffices to prove that their quadratic covariation with any $N \in \mathcal{M}_\ell$ are equal. Let $N \in \mathcal{M}_\ell$. By Theorem 4.2.8, $[(1_F H) \cdot M, N] = 1_F H \cdot [M,N] = 1_F (H \cdot [M,N])$, while Lemma 3.3.8 shows that we have $[1_F (H \cdot M), N] = 1_F [H \cdot M, N] = 1_F (H \cdot [M,N])$ and $[H \cdot 1_F M, N] = H \cdot [1_F M, N] = H \cdot 1_F [M,N] = 1_F (H \cdot [M,N])$. Thus, the quadratic covariation with N is equal to $1_F (H \cdot [M,N])$ for all three processes, and so Lemma 3.3.8 shows that $(1_F H) \cdot M = 1_F (H \cdot M) = H \cdot (1_F M)$, as desired.

Proof of (10). From what we already have shown, we find

$$\begin{aligned}(H \cdot X)^T &= H 1_{[\![0,T]\!]} \cdot X = H^T 1_{[\![0,T]\!]} \cdot X \\ &= K^T 1_{[\![0,T]\!]} \cdot X = K 1_{[\![0,T]\!]} \cdot X = (K \cdot X)^T,\end{aligned}$$

as desired. □

Our final lemma of this section shows that our construction of the stochastic integral coincides with the intuitive definition in the case of simple integrands.

Lemma 4.2.12. *Let X be a semimartingale, let $S \leq T$ be stopping times and let ξ be bounded and $\mathcal{F}_S$ measurable. Defining $H = \xi 1_{]\!]S,T]\!]}$, it holds that $H \in \mathfrak{I}$ and $(H \cdot X)_t = \xi(X^T - X^S)$.*

Proof. With $H = \xi 1_{]\!]S,T]\!]}$, we find by Lemma 4.2.1 that H is predictable. As it is also bounded, we obtain $H \in \mathfrak{I}$. Let $X = X_0 + M + A$ be a decomposition of X. Lemma 4.2.1 then yields $H \cdot M = \xi(M^T - M^S)$, and using the properties of ordinary Lebesgue integrals, we furthermore have $H \cdot A = \xi(A^T - A^S)$. Thus, $H \cdot X = \xi(X^T - X^S)$. □

4.3 Itô's formula

In this section, we will prove Itô's formula for semimartingales. In order to do so, we first prove some related results of general interest. We begin by demonstrating the dominated convergence theorem for stochastic integrals, and then obtain as moderately simple corollaries the limit characterisations of stochastic integrals and the quadratic covariation as well as the integration-by-parts formula. Applying these results, we may obtain Itô's formula.

Theorem 4.3.1. *Let X be a semimartingale and let $t \geq 0$ be some constant. Assume that $(H^n) \subseteq \mathfrak{I}$ and $H \in \mathfrak{I}$. Further assume that $|H^n 1_{[\![0,t]\!]}|$ and $|H 1_{[\![0,t]\!]}|$ are bounded by some K with $K \in \mathfrak{I}$. If H^n converges pointwise to H on $[\![0,t]\!]$, then*

$$\sup_{s \leq t} |(H^n \cdot X)_s - (H \cdot X)_s| \xrightarrow{P} 0.$$

Proof. As the stochastic integral does not depend on the initial value of the integrator, we may restrict our attention to the case where X_0 is zero.

Step 1. The case of $X \in \mathcal{M}^2$ and bounded integrands. We first consider the case where $X \in \mathcal{M}^2$ and H^n, H and K are bounded by some constant c. For convenience, we write M instead of X, and thus seek to show $\sup_{s \leq t} |(H^n \cdot M)_s - (H \cdot M)_s| \xrightarrow{P} 0$.

To this end, note that by Lemma 4.2.11, we have $[(H^n - H) \cdot M] = (H^n - H)^2 \cdot [M]$, and $E((H^n - H)^2 \cdot [M])_\infty \leq 4c^2 E[M]_\infty$, which is finite by Theorem 3.3.10. Therefore, we have $(H^n - H) \cdot M \in \mathcal{M}^2$. We will show that $((H^n - H) \cdot M)_\infty$ converges in $\mathcal{L}^2$ to zero, this will yield the desired result.

To this end, note that since $E[M]_\infty$ is finite, $[M]_\infty$ is almost surely finite and therefore the induced measures almost surely have finite mass. As a result, constants are almost surely integrable with respect to $[M]$. By Lebesgue's dominated convergence theorem, we then obtain

$$\lim_n \int_0^\infty (H^n_s - H_s)^2 \, \mathrm{d}[M]_s = \int_0^\infty \lim_n (H^n_s - H_s)^2 \, \mathrm{d}[M]_s = 0,$$

almost surely. Next, since $\int_0^\infty (H^n_s - H_s)^2 \, \mathrm{d}[M]_s \leq 4c^2[M]_\infty$, we can again use Lebesgue's dominated convergence theorem to obtain

$$\lim_n E \int_0^\infty (H^n_s - H_s)^2 \, \mathrm{d}[M]_s = E \lim_n \int_0^\infty (H^n_s - H_s)^2 \, \mathrm{d}[M]_s = 0.$$

As a result, $\lim_n E[(H^n - H) \cdot M]_\infty = \lim_n E((H^n - H)^2 \cdot [M])_\infty = 0$. Since we earlier noted that $(H^n - H) \cdot M \in \mathcal{M}^2$, we have $E[(H^n - H) \cdot M]_\infty = E((H^n - H) \cdot M)^2_\infty$ by Theorem 3.3.10, so we conclude that $((H^n - H) \cdot M)_\infty$ converges to zero in $\mathcal{L}^2$. By Theorem 1.3.3, we then obtain that $\sup_{t \geq 0} ((H^n - H) \cdot M)^*_\infty$ converges in $\mathcal{L}^2$ to zero, in particular we obtain that $\sup_{s \leq t} |(H^n \cdot M)_s - (H \cdot M)_s| \xrightarrow{P} 0$, as desired.

Step 2. The case of $X \in \mathcal{V}$ and bounded integrands. Now we consider the case where $X \in \mathcal{V}$, and continue to assume that H^n, H and K are bounded by a constant c. We write A instead of X and wish to show $\sup_{s \leq t} |(H^n \cdot A)_s - (H \cdot A)_s| \xrightarrow{P} 0$. By the properties of ordinary Lebesgue integrals, we have

$$\begin{aligned} \limsup_n \sup_{s \leq t} |(H^n \cdot A)_s - (H \cdot A)_s| &\leq \limsup_n \sup_{s \leq t} \int_0^s |H^n_u - H_u| \, |\mathrm{d}A_u| \\ &= \limsup_n \int_0^t |H^n_s - H_s| \, |\mathrm{d}A_s|. \end{aligned}$$

Since the measures induced by A have finite mass on compacts almost surely, the above is zero almost surely by Lebesgue's dominated convergence theorem, using the constant bound $2c$. Thus, $\sup_{s \leq t} |(H^n \cdot A)_s - (H \cdot A)_s|$ converges almost surely to zero, which implies $\sup_{s \leq t} |(H^n \cdot A)_s - (H \cdot A)_s| \xrightarrow{P} 0$, as was to be proven.

Step 3. The general case. Now consider a semimartingale X and processes (H^n), H and K in $\mathfrak{I}$ satisfying the conditions of the theorem. By Lemma 4.1.5, there exists a decomposition $X = M + A$ where $M \in \mathcal{M}^b_\ell$ and $A \in \mathcal{V}$. Let (T_n) be a localising sequence such that $K^{T_n} 1_{(T_n > 0)}$ is bounded and $M^{T_n} \in \mathcal{M}^b$. Then $|(H^n)^{T_n} 1_{(T_n > 0)} 1_{[\![0,t]\!]}| \leq K^{T_n} 1_{(T_n > 0)} 1_{[\![0,t]\!]}$ for all $n \geq 1$ and similarly $|H^{T_n} 1_{(T_n > 0)} 1_{[\![0,t]\!]}| \leq K^{T_n} 1_{(T_n > 0)} 1_{[\![0,t]\!]}$. By what we already have shown, it then holds that $\sup_{s \leq t} |((H^n)^{T_n} 1_{(T_n > 0)} \cdot M^{T_n})_s - (H^{T_n} 1_{(T_n > 0)} \cdot M^{T_n})_s| \xrightarrow{P} 0$ and $\sup_{s \leq t} |((H^n)^{T_n} 1_{(T_n > 0)} \cdot A^{T_n})_s - (H^{T_n} 1_{(T_n > 0)} \cdot A^{T_n})_s| \xrightarrow{P} 0$, which may be combined to obtain

$$\sup_{s \leq t} |((H^n)^{T_n} 1_{(T_n > 0)} \cdot X^{T_n})_s - (H^{T_n} 1_{(T_n > 0)} \cdot X^{T_n})_s| \xrightarrow{P} 0.$$

Now note that

$$1_{(T_n>t)} \sup_{s\le t} |(H^n \cdot X)_s - (H\cdot X)_s| = \sup_{s\le t} 1_{(T_n>t)}|(H^n\cdot X)_s - (H\cdot X)_s|,$$

and applying Lemma 4.2.11, we have

$$\begin{aligned} 1_{(T_n>t)}|(H^n\cdot X)_s - (H\cdot X)_s| &= 1_{(T_n>t)}|(H^n\cdot X)^{T_n}_s - (H\cdot X)^{T_n}_s| \\ &= 1_{(T_n>t)}|((H^n)^{T_n}\cdot X^{T_n})_s - (H^{T_n}\cdot X^{T_n})_s| \\ &= 1_{(T_n>t)}1_{(T_n>0)}|((H^n)^{T_n}\cdot X^{T_n})_s - (H^{T_n}\cdot X^{T_n})_s| \\ &\le |((H^n)^{T_n}1_{(T_n>0)}\cdot X^{T_n})_s - (H^{T_n}1_{(T_n>0)}\cdot X^{T_n})_s|. \end{aligned}$$

Combining these observations, we obtain $1_{(T_n>t)} \sup_{s\le t} |(H^n\cdot X)_s - (H\cdot X)_s| \xrightarrow{P} 0$. Applying Lemma A.3.1 to the sequence of variables $(\sup_{s\le t}|(H^n\cdot X)_s - (H\cdot X)_s|)_{n\ge1}$ and the sequence of sets $F_k = (T_k > t)$, the result follows. □

Next, we prove limit characterisations of the stochastic integral and the quadratic covariation, providing the integral interpretation of $H\cdot X$ and the quadratic covariation interpretation of $[X,Y]$. We first introduce some notation. Fix $t\ge 0$. We say that a finite increasing sequence $(t_0,\ldots,t_K)$ with $0=t_0\le\cdots\le t_K=t$ is a partition of $[0,t]$. We refer to $\max_{k\le K}|t_k - t_{k-1}|$ as the mesh of the partition.

Theorem 4.3.2. *Let $X\in\mathcal{S}$ and let H be adapted and càdlàg. Let $t\ge 0$ and assume that $(t^n_k)_{k\le K_n}$, $n\ge 1$, is a sequence of partitions of $[0,t]$ with mesh tending to zero. Then*

$$\sum_{k=1}^{K_n} H_{t^n_{k-1}}(X_{t^n_k} - X_{t^n_{k-1}}) \xrightarrow{P} (H_-\cdot X)_t.$$

Proof. First note that as H is adapted and càdlàg, H_- is adapted and càglàd, so $H_-\in\mathfrak{I}$ by Lemma 4.2.4 and the integral $H\cdot X$ is well-defined. To prove the result, we define the sequence of processes $H^n = H_0 1_{[\![0]\!]} + \sum_{k=1}^{K_n} H_{t^n_{k-1}} 1_{]\!]t^n_{k-1},t^n_k]\!]}$. As the mesh of the partitions converges to zero, we find that H^n converges pointwise to $H_- 1_{[\![0,t]\!]}$. Also note that as H is càdlàg and adapted, so is H^*, where $H^*_t = \sup_{s\le t}|H_s|$. In particular, $H^*_-\in\mathfrak{I}$ by Lemma 4.2.4, and both H^n and H_- are bounded by H^*_-. Therefore, by Theorem 4.3.1, we obtain $(H^n\cdot X)_t \xrightarrow{P} (H_-\cdot X)_t$.

However, $H_0 1_{[\![0]\!]}\cdot X$ is evanescent and so $\sum_{k=1}^{K_n} H_{t^n_{k-1}}(X_{t^n_k} - X_{t^n_{k-1}}) = (H^n\cdot X)_t$ by Lemma 4.2.12. Combining our conclusions, we obtain $\sum_{k=1}^{K_n} H_{t^n_{k-1}}(X_{t^n_k} - X_{t^n_{k-1}}) \xrightarrow{P} (H_-\cdot X)_t$. □

Theorem 4.3.3 (Integration-by-parts formula)**.** *Let X and Y be semimartingales. Let $t \geq 0$ and assume that $(t_k^n)_{k \leq K_n}$, $n \geq 1$, is a sequence of partitions of $[0, t]$ with mesh tending to zero. Then*

$$\sum_{k=1}^{K_n} (X_{t_k^n} - X_{t_{k-1}^n})(Y_{t_k^n} - Y_{t_{k-1}^n}) \xrightarrow{P} [X, Y]_t,$$

and the identity $X_t Y_t = X_0 Y_0 + (Y_- \cdot X)_t + (X_- \cdot Y)_t + [X, Y]_t$ holds.

Proof. We begin by considering a single semimartingale X and prove the two results

$$\sum_{k=1}^{K_n} (X_{t_k^n} - X_{t_{k-1}^n})^2 \xrightarrow{P} [X, X]_t \text{ and } X_t^2 = X_0^2 + 2(X_- \cdot X)_t + [X]_t.$$

We begin by assuming that $X = X_0 + M + A$, where $M \in \mathcal{M}^b$ and $A \in \mathcal{V}^i$. First note that

$$X_t^2 - X_0^2 = \sum_{k=1}^{K_n} (X_{t_k^n}^2 - X_{t_{k-1}^n}^2) = 2 \sum_{k=1}^{K_n} X_{t_{k-1}^n}(X_{t_k^n} - X_{t_{k-1}^n}) + \sum_{k=1}^{K_n} (X_{t_k^n} - X_{t_{k-1}^n})^2.$$

Since X is càdlàg and adapted, $\sum_{k=1}^{K_n} X_{t_{k-1}^n}(X_{t_k^n} - X_{t_{k-1}^n}) \xrightarrow{P} (X_- \cdot X)_t$ by Theorem 4.3.2, and therefore $\sum_{k=1}^{K_n} (X_{t_k^n} - X_{t_{k-1}^n})^2 \xrightarrow{P} X_t^2 - X_0^2 - 2(X_- \cdot X)_t$. Our proof of the present case now proceeds in three steps. Firstly, we argue that $X^2 - X_0^2 - 2(X_- \cdot X)$ has paths of finite variation. Secondly, we argue that $X^2 - X_0^2 - 2(X_- \cdot X) - [X]$ is continuous with paths of finite variation. Thirdly, we prove that this process is $\mathcal{M}_\ell$ and obtain the desired results from this. After this, we consider the remaining cases.

Step 1. Proof that $X^2 - X_0^2 - 2(X_- \cdot X) \in \mathcal{V}$. We wish to argue that $X^2 - X_0^2 - 2(X_- \cdot X)$ is almost surely increasing. To this end, let $0 \leq p \leq q$ be two elements of $\mathbb{D}_+$. There exists $j \geq 1$ and naturals $n_p \leq n_q$ such that $p = n_p 2^{-j}$ and $q = n_q 2^{-j}$. Consider the particular partitions of $[0, p]$ and $[0, q]$ given by putting $p_k^n = k 2^{-(n+j)}$ for $k \leq n_p 2^n$ and $q_k^n = k 2^{-(n+j)}$ for $k \leq n_q 2^n$, respectively. Using Lemma A.3.2 and the convergence result just proven, we then obtain

$$\begin{aligned} X_p^2 - X_0^2 - 2(X_- \cdot X)_p &= \lim_n \sum_{k=1}^{n_p 2^n} (X_{p_k^n} - X_{p_{k-1}^n})^2 \\ &\leq \lim_n \sum_{k=1}^{n_q 2^n} (X_{q_k^n} - X_{q_{k-1}^n})^2 \\ &= X_q^2 - X_0^2 - 2(X_- \cdot X)_q, \end{aligned}$$

almost surely, where the limits are in probability. As $\mathbb{D}_+$ is countable and dense in $\mathbb{R}_+$, we conclude that $X^2 - X_0^2 - 2(X_- \cdot X)$ is almost surely increasing. By picking a particular modification of $X_- \cdot X$, we may assume that $X^2 - X_0^2 - 2(X_- \cdot X)$ is increasing. In particular, $X^2 - X_0^2 - 2(X_- \cdot X) \in \mathcal{V}$.

Step 2. Proof that $X^2 - X_0^2 - 2(X_- \cdot X) - [X]$ is continuous and in $\mathcal{V}$. By direct calculation, we have

$$\begin{aligned} \Delta(X^2 - 2X_- \cdot X)_t &= X_t^2 - X_{t-}^2 - 2X_{t-}\Delta X_t = X_t^2 - X_{t-}^2 - 2X_{t-}X_t + 2X_{t-}^2 \\ &= X_t - 2X_tX_{t-} + X_{t-}^2 = (\Delta X)_t^2 = \Delta[X]. \end{aligned}$$

As we know that $[X] \in \mathcal{V}$ and $X^2 - X_0^2 - 2(X_- \cdot X) \in \mathcal{V}$, we find that $X^2 - X_0^2 - 2(X_- \cdot X) - [X]$ is a continous process in $\mathcal{V}$. Therefore, if we can show that it is a local martingale, Theorem 3.1.9 will yield $X^2 = X_0 + 2(X_- \cdot X) + [X]$.

Step 3. Conclusion. We wish to argue that $X^2 - X_0^2 - 2(X_- \cdot X) - [X]$ is in $\mathcal{M}_\ell$. First note that with $Y = X - X_0$, we have $Y = M + A$ and, using that $X = Y + X_0$,

$$\begin{aligned} X_t^2 - X_0^2 - 2(X_- \cdot X)_t - [X]_t &= Y_t^2 + 2Y_tX_0 - 2(X_- \cdot X)_t - [Y]_t \\ &= Y_t^2 + 2Y_tX_0 - 2X_0(X_t - X_0) - 2(Y_- \cdot X)_t - [Y]_t \\ &= Y_t^2 - 2(Y_- \cdot X)_t - [Y]_t \\ &= Y_t^2 - 2(Y_- \cdot Y)_t - [Y]_t. \end{aligned}$$

Recalling that $A^2 - 2A_- \cdot A - [A]$ is evanescent by Lemma A.2.13, we then obtain

$$X^2 - X_0^2 - X_- \cdot X - [X] = (M + A)^2 - (M + A)_- \cdot (M + A) - [M + A] = N^m + 2N^x,$$

where $N^m = M^2 - 2M_- \cdot M - [M]$ and $N^x = MA - M_- \cdot A - A_- \cdot M - [M, A]$. We claim that both of these processes are in $\mathcal{M}_\ell$. As $M^2 - [M]$ is in $\mathcal{M}_\ell$ by Theorem 3.3.4 and $M_- \cdot M$ is in $\mathcal{M}_\ell$ by Theorem 4.2.8, it is immediate that N^m is in $\mathcal{M}_\ell$. Regarding the second process, first note that $A_- \cdot M$ is a local martingale since $A_- \in \mathfrak{I}$ by Lemma 4.2.4 and $M \in \mathcal{M}_\ell$. Furthermore, note that by the properties of ordinary Lebesgue integrals, we have

$$\begin{aligned} M_tA_t - (M_- \cdot A)_t - [M, A]_t &= M_tA_t - \int_0^t M_{s-} \,\mathrm{d}A_s - \sum_{0<s\leq t} \Delta M_s \Delta A_s \\ &= M_tA_t - \int_0^t M_{s-} \,\mathrm{d}A_s - \int_0^t \Delta M_s \,\mathrm{d}A_s \\ &= M_tA_t - \int_0^t M_s \,\mathrm{d}A_s, \end{aligned}$$

which is in $\mathcal{M}^u$ by Lemma 3.3.3. Combining our findings, $X^2 - X_0 - 2(X_- \cdot X) - [X]$ is a continuous element of $\mathbf{fv}\mathcal{M}_\ell$, so by Theorem 3.1.9 it is evanescent. We conclude that the integration-by-parts formula $X^2 = X_0 + 2(X_- \cdot X) + [X]$ holds. From what was previously shown, we then also obtain $\sum_{k=1}^{K_n}(X_{t_k^n} - X_{t_{k-1}^n})^2 \xrightarrow{P} [X]_t$.

Step 4. Remaining cases. We now consider the result for a single general semimartingale X. By Lemma 4.1.5, there exists a decomposition $X = X_0 + M + A$ where $M \in \mathcal{M}_\ell^b$ and $A \in \mathcal{V}_\ell^i$. Let (T_n) be a localising sequence such that M^{T_n} and $A^{T_n} \in \mathcal{V}^i$. From what we already have shown, it holds that

$$1_{(T_n>t)}\sum_{k=1}^{K_n}(X_{t_k^n} - X_{t_{k-1}^n})^2 = 1_{(T_n>t)}\sum_{k=1}^{K_n}(X^{T_n}_{t_k^n} - X^{T_n}_{t_{k-1}^n})^2 \xrightarrow{P} 1_{(T_n>t)}[X^{T_n}]_t,$$

and by Lemma 4.1.13, this yields $1_{(T_n>t)}\sum_{k=1}^{K_n}(X_{t_k^n} - X_{t_{k-1}^n})^2 \xrightarrow{P} 1_{(T_n>t)}[X]_t$. Applying Lemma A.3.1, we conclude $\sum_{k=1}^{K_n}(X_{t_k^n} - X_{t_{k-1}^n})^2 \xrightarrow{P} [X]_t$. Next, using Lemma 4.2.11, we also have

$$\begin{aligned}(X^2)^{T_n}_t &= (X^{T_n})^2_t = (X^{T_n})^2_0 + 2((X^{T_n})_- \cdot X^{T_n})_t + [X^{T_n}]_t \\ &= X_0^2 + 2(X_-^{T_n} \cdot X)^{T_n}_t + [X^{T_n}]_t = X_0^2 + 2(X_- \cdot X)^{T_n}_t + [X]^{T_n}_t,\end{aligned}$$

so letting n tend to infinity, we obtain $X_t^2 = X_0 + 2(X_- \cdot X)_t + [X]_t$. This concludes the proof of the theorem in the case of a single semimartingale.

It remains to consider the case of two semimartingales X and Y. Define two processes $Z = X + Y$ and $W = X - Y$, we then have $Z + W = 2X$ and $Z - W = 2Y$, yielding

$$\begin{aligned}(Z_{t_k^n} - Z_{t_{k-1}^n})^2 - (W_{t_k^n} - W_{t_{k-1}^n})^2 &= (2X_{t_k^n} - 2X_{t_{k-1}^n})(2Y_{t_k^n} - 2Y_{t_{k-1}^n}) \\ &= 4(X_{t_k^n} - X_{t_{k-1}^n})(Y_{t_k^n} - Y_{t_{k-1}^n}),\end{aligned}$$

and we know from our previous results that $\sum_{k=1}^{K_n}(Z_{t_k^n} - Z_{t_{k-1}^n})^2$ converges in probability to $[Z]_t$ and that $\sum_{k=1}^{K_n}(W_{t_k^n} - W_{t_{k-1}^n})^2$ converges in probability to $[W]_t$. By Lemma 4.1.13, we have $[Z]_t - [W]_t = [X+Y]_t - [X-Y]_t = 4[X,Y]_t$ almost surely, so collecting our results, we finally conclude $\sum_{k=1}^{K_n}(X_{t_k^n} - X_{t_{k-1}^n})(Y_{t_k^n} - Y_{t_{k-1}^n}) \xrightarrow{P} [X,Y]_t$, as desired. Analogously, we find

$$\begin{aligned}4X_tY_t &= Z_t^2 - W_t^2 \\ &= Z_0^2 - W_0^2 + 2(Z \cdot Z)_t - 2(W \cdot W) + [Z]_t - [W]_t \\ &= 4X_0Y_0 + 2((X+Y) \cdot (X+Y))_t - 2((X-Y) \cdot (X-Y))_t + 4[X,Y]_t \\ &= 4X_0Y_0 + 4(X \cdot Y)_t + 4(Y \cdot X)_t + 4[X,Y]_t,\end{aligned}$$

yielding the integration-by-parts formula in the general case. This concludes the proof. □

Lemma 4.3.4. *Let X and Y be semimartingales and let H be adapted and càdlàg. Consider $t \geq 0$ and assume that $(t^n_k)_{k \leq K_n}$, $n \geq 1$, is a sequence of partitions of $[0,t]$ with mesh tending to zero. Then*

$$\sum_{k=1}^{K_n} H_{t^n_{k-1}}(X_{t^n_k} - X_{t^n_{k-1}})(Y_{t^n_k} - Y_{t^n_{k-1}}) \xrightarrow{P} (H_- \cdot [X,Y])_t.$$

Proof. As in the proof of Theorem 4.3.3, by polarization, it will suffice to consider a single semimartingale X and prove $\sum_{k=1}^{K_n} H_{t^n_{k-1}}(X_{t^n_k} - X_{t^n_{k-1}})^2 \xrightarrow{P} (H_- \cdot [X])_t$. Also note that we may assume without loss of generality that X has initial value zero. To prove the result in this case, note that from Theorem 4.3.3, $[X]_t = X_t^2 - 2(X_- \cdot X)_t$, so that using Lemma 4.2.11, we find $H_- \cdot [X] = H_- \cdot X^2 - 2H_- \cdot (X_- \cdot X) = H_- \cdot X^2 - 2H_- X_- \cdot X$, where $X^2 \in \mathcal{S}$ since $X^2 = X_0 + 2(X_- \cdot X) + [X]$. On the other hand,

$$\sum_{k=1}^{K_n} H_{t^n_{k-1}}(X_{t^n_k} - X_{t^n_{k-1}})^2 = \sum_{k=1}^{K_n} H_{t^n_{k-1}}(X^2_{t^n_k} - X^2_{t^n_{k-1}}) - 2\sum_{k=1}^{K_n} H_{t^n_{k-1}} X_{t^n_{k-1}}(X_{t^n_k} - X_{t^n_{k-1}}),$$

so that two applications of Theorem 4.3.2 yield the result. □

With the above results in hand, we are now ready to prove Itô's formula. By $C^2(\mathbb{R}^p)$, we denote the set of mappings $f : \mathbb{R}^p \to \mathbb{R}$ such that all second-order partial derivatives of f exist and are continuous. Also, for any open set U in $\mathbb{R}^p$, we denote by $C^2(U)$ the set of mappings $f : U \to \mathbb{R}$ with the same property. We say that a process X with values in $\mathbb{R}^p$ is a p-dimensional semimartingale if each of its coordinate processes X^i, where $X_t = (X^1_t, \ldots, X^p_t)$, is a semimartingale.

Theorem 4.3.5 (Itô's formula)**.** *Let X be a p-dimensional semimartingale and consider $f \in C^2(\mathbb{R}^p)$. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(X_{s-})\,\mathrm{d}X^i_s + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})\,\mathrm{d}[X^i, X^j]_s + \eta_t,$$

up to indistinguishability, where

$$\eta_t = \sum_{0<s\leq t} f(X_s) - f(X_{s-}) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{s-})\Delta X^i_s - \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})\Delta X^i_s \Delta X^j_s.$$

Here, almost surely, the sum defining η is absolutely convergent for all $t \geq 0$.

Proof. We first argue that the sum defining η converges absolutely. First consider the case where X takes its values in a compact set. Fix $t \geq 0$. By Theorem A.1.22, we then have, with r_2^{ij} denoting the remainder function given in the statement of Theorem A.1.22,

$$\begin{aligned}
&\left|f(X_s)-f(X_{s-})-\sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i-\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})\Delta X_s^i \Delta X_s^j\right| \\
=\ &\left|\sum_{i=1}^p\sum_{j=1}^p r_2^{ij}(X_s,X_{s-})(\Delta X_s)^2\right|.
\end{aligned}$$

By Theorem A.1.22, $r_2^{ij}(X_s, X_{s-})$ is bounded by the values of $\frac{\partial^2 f}{\partial x_i \partial x_j}$ on the line segment from X_s to X_{s-}. As X takes its values in a compact set, we obtain that there is $C>0$ such that $|r_2^{ij}(X_s,X_{s-})| \le C$ for all $i,j \le p$. Thus, the above sum is bounded by $Cp^2(\Delta X_s)^2$. Therefore, by Lemma 4.1.11, almost surely, the sum converges absolutely for all $t \ge 0$.

Now consider the general case. Using Lemma 4.1.8, let (T_n) be a localising sequence such that X^{T_n-} is bounded. By the above argument, we then obtain that almost surely, whenever $T_n > 0$, the sum converges absolutely for all $0 \le t < T_n$. Letting n tend to infinity, it follows that almost surely, the sum defining η is absolutely convergent for all $t \ge 0$.

Next, we prove the formula for $f(X_t)$. Fix $t \ge 0$ and let $t_k^n = kt2^{-n}$. Applying a telescoping sum, we obtain $f(X_t)-f(X_0)=\sum_{k=1}^{2^n} f(X_{t_k^n})-f(X_{t_{k-1}^n})$ and may use Theorem A.1.22 to obtain $f(X_t)=f(X_0)+S_t^n+T_t^n+R_t^n$ where

$$\begin{aligned}
S_t^n &= \sum_{i=1}^p\sum_{k=1}^{2^n}\frac{\partial f}{\partial x_i}(X_{t_{k-1}^n})(X_{t_k^n}^i-X_{t_{k-1}^n}^i)\\
T_t^n &= \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p\sum_{k=1}^{2^n}\frac{\partial^2 f}{\partial x_i\partial x_j}(X_{t_{k-1}^n})(X_{t_k^n}^i-X_{t_{k-1}^n}^i)(X_{t_k^n}^j-X_{t_{k-1}^n}^j)\\
R_t^n &= \sum_{i=1}^p\sum_{j=1}^p\sum_{k=1}^{2^n} r_2^{ij}(X_{t_{k-1}^n},X_{t_k^n})(X_{t_k^n}^i-X_{t_{k-1}^n}^i)(X_{t_k^n}^j-X_{t_{k-1}^n}^j),
\end{aligned}$$

and $r_2^{ij}(x,y)$ is the remainder function from Theorem A.1.22. By Theorem 4.3.2, S_t^n converges in probability to $\sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(X_{s-})\,\mathrm{d}X_s^i$, and by Lemma 4.3.4, T_t^n converges in probability to $\frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p\int_0^t \frac{\partial f}{\partial x_i \partial x_j}(X_{s-})\,\mathrm{d}[X^i,X^j]_s$. Therefore, it will suffice to show that the remainder term R_t^n converges in probability to η_t. Note that while we have no guarantee that r_2^{ij} is measurable, we know that R_t^n is always measurable, since $R_t^n = f(X_t)-f(X_0)-S_t^n-T_t^n$, and so the proposition that R_t^n converges in probability to η_t is well-defined.

To prove $R_t^n \xrightarrow{P} \eta_t$, first consider the case where X takes its values in a compact set. In particular, all second-order partial derivatives of f are uniformly continuous on the range of

X. Let $\varepsilon > 0$. Also fix $\gamma > 0$, and pick $\delta > 0$ parrying γ for this uniform continuity of the second-order partial derivatives of f, and assume without loss of generality that $\delta \leq \gamma$. Let $A_\delta = \{s \in [0,t] \mid |\Delta X_s| \geq \delta/4\}$. Note that as X is càdlàg, $A_\delta(\omega)$ is a finite set for all ω. Define $I_n = \{k \leq 2^n \mid A_\delta \cap (t^n_{k-1}, t^n_k] = \emptyset\}$ and $J_n = \{1, \ldots, 2^n\} \setminus I_n$, and define

$$\begin{aligned}
(R^{(J)})^n_t &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k \in J_n} r_2^{ij}(X_{t^n_{k-1}}, X_{t^n_k})(X^i_{t^n_k} - X^i_{t^n_{k-1}})(X^j_{t^n_k} - X^j_{t^n_{k-1}}) \quad \text{and} \\
(R^{(I)})^n_t &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k \in I_n} r_2^{ij}(X_{t^n_{k-1}}, X_{t^n_k})(X^i_{t^n_k} - X^i_{t^n_{k-1}})(X^j_{t^n_k} - X^j_{t^n_{k-1}}).
\end{aligned}$$

Furthermore, define

$$\eta(A_\delta)_t = \sum_{s \in A_\delta} f(X_s) - f(X_{s-}) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{s-})\Delta X^i_s - \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})\Delta X^i_s \Delta X^j_s,$$

and analogously, $\eta(A^c_\delta)_t$. As A_δ is finite, we have $(R^{(J)})^n_t \xrightarrow{\text{a.s.}} \eta(A_\delta)_t$. Also noting that $A_\gamma \subseteq A_\delta$ as $\delta \leq \gamma$, we get $A^c_\delta \subseteq A^c_\gamma$ and thus obtain

$$\begin{aligned}
& \limsup_{n \to \infty} P(|R^n_t - \eta_t| \geq \varepsilon) \\
\leq\; & \limsup_{n \to \infty} P(|(R^{(J)})^n_t - \eta(A_\delta)_t| \geq \varepsilon/2) + P(|(R^{(I)})^n_t - \eta(A^c_\delta)_t| \geq \varepsilon/2) \\
\leq\; & P(|\eta(A^c_\gamma)_t| \geq \varepsilon/4) + \limsup_{n \to \infty} P(|(R^{(I)})^n_t| \geq \varepsilon/4).
\end{aligned}$$

Next, we bound the limes superior in the above. To this end, recall that as $|2xy| \leq x^2 + y^2$, we find for any $i, j \leq p$ that

$$\begin{aligned}
& \left| \sum_{k \in I_n} r_2^{ij}(X_{t^n_{k-1}}, X_{t^n_k})(X^i_{t^n_k} - X^i_{t^n_{k-1}})(X^j_{t^n_k} - X^j_{t^n_{k-1}}) \right| \\
\leq\; & \frac{1}{2}\left(\max_{k \in I_n} |r_2^{ij}(X_{t^n_{k-1}}, X_{t^n_k})| \right) \left(\sum_{k=1}^{2^n} (X^i_{t^n_k} - X^i_{t^n_{k-1}})^2 + \sum_{k=1}^{2^n} (X^j_{t^n_k} - X^j_{t^n_{k-1}})^2 \right),
\end{aligned}$$

Now note that by Theorem A.1.22, there is a mapping $\xi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ such that $r_2^{ij}(X_{t^n_{k-1}}, X_{t^n_k}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi(X_{t^n_{k-1}}, X_{t^n_k})) - \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t^n_{k-1}})$, where $\xi(x,y)$ always is on the line segment between x and y. In particular, we have

$$\max_{k \in I_n} |r_2^{ij}(X_{t^n_{k-1}}, X_{t^n_k})| \leq \max_{k \in I_n} \sup_{t \in [0,1]} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t^n_{k-1}} + t(X_{t^n_k} - X_{t^n_{k-1}})) - \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t^n_{k-1}}) \right|,$$

and the latter is measurable, since by right-continuity, the supremum may be reduced to a countable one. Now, by Lemma A.2.5, it holds that

$$\limsup_{n \to \infty} \max_{k \in I_n} \sup_{t^n_{k-1} \leq r, s \leq t^n_k} |X_s - X_r| \leq 3 \sup_{s \in [0,t] \setminus A_\delta} |\Delta f(X_s)| < \delta,$$

so for n large enough, depending on ω, $\max_{k\in I_n}|r_2^{ij}(X_{t_{k-1}^n},X_{t_k^n})|\le\gamma$. For each ω, let $N(\omega)$ be the first n such that this bound holds. Then, for $n\ge N$, we obtain the bound

$$(R^{(I)})_t^n \le \frac{\gamma}{2}\sum_{i=1}^p\sum_{j=1}^p\left(\sum_{k=1}^{2^n}(X_{t_k^n}^i-X_{t_{k-1}^n}^i)^2+\sum_{k=1}^{2^n}(X_{t_k^n}^j-X_{t_{k-1}^n}^j)^2\right),$$

and by Theorem 4.3.3, the latter converges to $\frac{\gamma}{2}\sum_{i=1}^p\sum_{j=1}^p[X^i]_t+[X^j]_t$ in probability. We are now ready to bound $\limsup_{n\to\infty}P(|(R^{(I)})_t^n|\ge\varepsilon/4)$. By dominated convergence theorem, we obtain $\limsup_{n\to\infty}P(|(R^{(I)})_t^n|\ge\varepsilon/4)\le\limsup_{n\to\infty}P((n\ge N)\cap(|(R^{(I)})_t^n|\ge\varepsilon/4))$, where

$$\begin{aligned}
&\limsup_{n\to\infty}P((n\ge N)\cap(|(R^{(I)})_t^n|\ge\varepsilon/4))\\
\le\;&\limsup_{n\to\infty}P\left(\left|\frac{\gamma}{2}\sum_{i=1}^p\sum_{j=1}^p\left(\sum_{k=1}^{2^n}(X_{t_k^n}^i-X_{t_{k-1}^n}^i)^2+\sum_{k=1}^{2^n}(X_{t_k^n}^j-X_{t_{k-1}^n}^j)^2\right)\right|\ge\frac{\varepsilon}{4}\right)\\
\le\;&P\left(\left|\frac{\gamma}{2}\sum_{i=1}^p\sum_{j=1}^p[X^i]_t+[X^j]_t\right|\ge\frac{\varepsilon}{8}\right).
\end{aligned}$$

All in all, we have now shown that

$$\limsup_{n\to\infty}P(|R_t^n-\eta_t|\ge\varepsilon)\le P(|\eta(A_\gamma^c)_t|\ge\varepsilon/4)+P\left(\left|\frac{\gamma}{2}\sum_{i=1}^p\sum_{j=1}^p[X^i]_t+[X^j]_t\right|\ge\frac{\varepsilon}{8}\right),$$

for all $\varepsilon>0$ and for all $\gamma>0$. Now note that A_γ^c decreases as γ decreases, and we have $\cap_{\gamma>0}A_\gamma^c=\{s\in[0,t]\mid\Delta X_s=0\}$. The dominated convergence theorem then yields that almost surely, $\eta(A_\gamma^c)_t$ converges to zero as γ tends to zero. Likewise, $\frac{\gamma}{2}\sum_{i=1}^p\sum_{j=1}^p[X^i]_t+[X^j]_t$ converges to zero as γ tends to zero. Letting γ tend to zero in the above, we then obtain $\limsup_{n\to\infty}P(|R_t^n-\eta_t|\ge\varepsilon)=0$, so $R_t^n\xrightarrow{P}\eta_t$ and the proof is complete in the case where X takes its values in a compact set.

It remains to show $R_t^n\xrightarrow{P}\eta_t$ in the case of a general X. To this end, define a localising sequence (T_m) by putting $T_m=\inf\{t\ge0\mid|X_t|>m\}$. By Lemma 4.1.8, $X^{T_m-}1_{(T_m>0)}$ is bounded by m. Fix $m\ge1$ and put $Y^m=X^{T_m-}1_{(T_m>0)}$, Y^m then takes its values in the compact set $[-m,m]^p$. Note that on $(T_m>t)$, we have

$$R_t^n=\sum_{i=1}^p\sum_{j=1}^p\sum_{k=1}^{2^n}r_2^{ij}(Y_{t_{k-1}^n}^m,Y_{t_k^n}^m)((Y^m)_{t_k^n}^i-(Y^m)_{t_{k-1}^n}^i)((Y^m)_{t_k^n}^j-(Y^m)_{t_{k-1}^n}^j)$$

and

$$\eta_t=\sum_{0<s\le t}f(Y_s^m)-f(Y_{s-}^m)-\sum_{i=1}^n\frac{\partial f}{\partial x_i}(Y_{s-}^m)\Delta(Y^m)_s^i-\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n\frac{\partial^2 f}{\partial x_i\partial x_j}(Y_{s-}^m)\Delta(Y^m)_s^i\Delta(Y^m)_s^j.$$

Therefore, by what we already have shown, $1_{(T_m>t)}R^n_t \xrightarrow{P} 1_{(T_m>t)}\eta_t$. Applying Lemma A.3.1, we obtain $R^n_t \xrightarrow{P} \eta_t$.

We have now proved that Itô's formula for a fixed $t \geq 0$ holds almost surely. As the processes on both sides of the formula are càdlàg and $t \geq 0$ was arbitrary, Lemma 1.1.7 shows that we have equality up to indistinguishability. This proves the theorem. □

In practical applications, it will occasionally be necessary to apply Itô's formula in cases where function f only is defined on some open set. The following corollary shows that Itô's formula also holds in this case.

Corollary 4.3.6. *Let U be an open set in $\mathbb{R}^p$, let X be a p-dimensional continuous semimartingale taking its values in U and let $f : U \to \mathbb{R}$ be C^2. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial x_i}(X_{s-})\,\mathrm{d}X^i_s + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})\,\mathrm{d}[X^i, X^j]_s + \eta_t,$$

up to indistinguishability, where

$$\eta_t = \sum_{0<s\leq t} f(X_s) - f(X_{s-}) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{s-})\Delta X^i_s - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})\Delta X^i_s \Delta X^j_s.$$

Proof. Let $\|\cdot\|$ be some norm on $\mathbb{R}^p$ and let $d(x,y) = \|x-y\|$. Define the set U_m by putting $U_m = \{x \in \mathbb{R}^p \mid d(x, U^c) < \frac{1}{m}\}$. Put $F_m = U_m^c$, then $F_m = \{x \in \mathbb{R}^p \mid d(x, U^c) \geq \frac{1}{m}\}$. As $x \mapsto d(x, U^c)$ is continuous, U_m is open and F_m is closed. Put $T_m = \inf\{t \geq 0 \mid X_t \in U_m\}$. As U_m is open, Lemma 1.1.13 shows that T_m is a stopping time. As (U_m) is decreasing, (T_m) is increasing. We wish to argue that (T_m) tends to infinity almost surely and that on $(T_m > 0)$, X^{T_m-} takes its values in F_m.

To prove that (T_m) tends to infinity almost surely, assume that there is ω such that $T_m(\omega)$ has a finite limit $T(\omega)$. By construction, there is for each $\varepsilon > 0$ a $t \in [T_m(\omega), T_m(\omega)+\varepsilon]$ such that $X_t \in U_m$, meaning that $d(X_t, U^c) < \frac{1}{m}$. Applying right-continuity, this yields $d(X_{T_m}(\omega), U^c) \leq \frac{1}{m}$. And by left-continuity, $d(X_T(\omega), U^c) = \lim_m d(X_{T_m}(\omega), U^c) = 0$, implying $X_T(\omega) \in U^c$, a contradiction. We conclude that (T_m) tends almost surely to infinity. To show that on $(T_m > 0)$, X^{T_m-} takes its values in F_m, we note that on this set, $X_t \notin U_m$ for $t < T_m$, so $X_t \in F_m$ for $t < T_m$, and by left-continuity of X and closedness of F_m, $X_{T_m-} \in F_m$ as well. Thus, X^{T_m-} takes its values in F_m on $(T_m > 0)$.

Now let m be so large that F_m is nonempty, this is possible as $U = \cup_{n=1}^{\infty} F_m$ and U is nonempty because X takes its values in U. Let y_m be some point in F_m. Define the process Y^m by putting $(Y^m)^i_t = 1_{(T_m>0)}(X^i)^{T_m-}_t + y^i_m 1_{(T_m=0)}$. Y^m is then a p-dimensional continuous semimartingale taking its values in F_m. Now, by Lemma A.1.23, there is a C^2 mapping $g_m : \mathbb{R}^p \to \mathbb{R}$ such that g_m and f agree on F_m. By Theorem 4.3.5, Itô's formula holds using Y^m and g_m, and as g_m and f agree on F_m, we obtain

$$\begin{aligned} f(Y^m_t) &= f(Y^m_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(Y^m_s)\,\mathrm{d}(Y^m)^i_s \\ &\quad + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Y^m_s)\,\mathrm{d}[(Y^m)^i,(Y^m)^j]_s + \eta^m_t, \end{aligned}$$

up to indistinguishability, where η^m is the jump sum as given in Theorem 4.3.5. We wish to argue that as m tends to infinity, all terms in the above converge to the corresponding terms with Y^m exchanged by X. Consider the first-order terms. For any $i \le p$, we may use Lemma 4.2.11 to obtain

$$\begin{aligned} 1_{(T_m>t)}\int_0^t \frac{\partial f}{\partial x_i}(Y^m_s)\,\mathrm{d}(Y^m)^i_s &= 1_{(T_m>t)}\int_0^t \frac{\partial f}{\partial x_i}(Y^m_s)\,\mathrm{d}(X^i)^{T_m}_s \\ &= 1_{(T_m>t)}\int_0^t 1_{(T_m>0)}\frac{\partial f}{\partial x_i}(X^{T_m}_s)\,\mathrm{d}(X^i)^{T_m}_s \\ &= 1_{(T_m>t)}\int_0^t \frac{\partial f}{\partial x_i}(X^{T_m}_s)\,\mathrm{d}(X^i)^{T_m}_s \\ &= 1_{(T_m>t)}\int_0^t \frac{\partial f}{\partial x_i}(X_s)\,\mathrm{d}X^i_s, \end{aligned}$$

and with an application of Lemma 4.1.13, the analogous statement is obtained for the second-order terms. Also, $1_{(T_m>t)}f(Y^m_t) = 1_{(T_m>t)}f(X_t)$ and $1_{(T_m>t)}f(Y^m_0) = 1_{(T_m>t)}f(X_0)$. All in all, we conclude that Itô's formula holds almost surely at time $t \ge 0$ on $(T_m > t)$, and letting m tend to infinity, we obtain that the formula holds at any time $t \ge 0$. By Lemma 1.1.7, the result holds up to indistinguishability and the proof is concluded. □

4.4 Exercises

Exercise 4.4.1. *Let $X \in \mathcal{S}$. We say that X is a special semimartingale and write $X \in \mathcal{S}_p$ if there exists a decomposition $X = X_0 + M + A$ where $M \in \mathcal{M}_\ell$ and $A \in \mathcal{V}$ is predictable. Show that if $X \in \mathcal{S}_p$, then the decomposition of X into its local martingale and predictable finite variation parts is unique up to evanescence.*

Exercise 4.4.2. *Let $X \in \mathcal{S}$ and assume that X has initial value zero. Show that $X \in \mathcal{S}_p$ if and only if $X^* \in \mathcal{A}^i_\ell$, where we define the process X^* by putting $X^*_t = \sup_{s\le t}|X_s|$.*

Exercise 4.4.3. *Let $X \in \mathcal{S}$. Show that $X \in \mathcal{S}_p$ if and only if there exists a localising sequence (T_n) such that $X^{T_n} \in \mathcal{S}_p$ for all $n \geq 1$.*

Exercise 4.4.4. *Let $X \in \mathcal{S}$. Show that X is predictable if and only if there exists a decomposition $X = X_0 + M + A$ where $M \in \mathcal{M}_\ell$ is almost surely continuous and $A \in \mathcal{V}$ with A predictable.*

Exercise 4.4.5. *Let $X \in \mathcal{S}$. Show that X has a decomposition of the form $X = X_0 + M + A$ with $M \in \mathcal{M}_\ell$ almost surely continuous and $A \in \mathcal{V}$ if and only if for all $t \geq 0$, $\sum_{0<s\leq t} |\Delta X_s|$ is almost surely convergent.*

Exercise 4.4.6. *Let $X \in \mathcal{S}$. Show that X is continuous if and only if there exists a decomposition $X = X_0 + M + A$ where $M \in \mathbf{c}\mathcal{M}_\ell$ and $A \in \mathcal{V}$ is continuous.*

Exercise 4.4.7. *Let $M \in \mathcal{M}_\ell$ and let T be a predictable stopping time. Show that M^{T-} is an element of $\mathcal{M}_\ell$.*

Exercise 4.4.8. *Let $M \in \mathcal{M}_\ell$ and let $H \in \mathfrak{I}$. Show that $H \cdot M \in \mathcal{M}^2$ if and only if $E\int_0^\infty H_s^2 \,\mathrm{d}[M]_s$ is finite, and in the affirmative, $E(H \cdot M)_\infty^2 = E\int_0^\infty H_s^2 \,\mathrm{d}[M]_s$.*

Exercise 4.4.9. *Let $M \in \mathcal{M}_\ell$ and let $H \in \mathfrak{I}$. Show that if $E\int_0^t H_s^2 \,\mathrm{d}[M]_s$ is finite for all $t \geq 0$, then $H \cdot M \in \mathcal{M}$ with $E(H \cdot M)_t^2 = E\int_0^t H_s^2 \,\mathrm{d}[M]_s$ for all $t \geq 0$.*

Exercise 4.4.10. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and let $H \in \mathfrak{I}$. Show that $H \cdot W$ is in $\mathbf{c}\mathcal{M}^2$ if and only if $E\int_0^\infty H_s^2 \,\mathrm{d}s$ is finite. Show that if it holds that for any $t \geq 0$, $E\int_0^t H_s^2 \,\mathrm{d}s$ is finite, then $H \cdot W$ is in $\mathbf{c}\mathcal{M}$ and $E(H \cdot W)_t^2 = E\int_0^t H_s^2 \,\mathrm{d}s$ for all $t \geq 0$.*

Exercise 4.4.11. *Let $A \in \mathcal{V}_\ell^i$ and let $H \in \mathfrak{I}$. Show that the compensator of $\int_0^t H_s \,\mathrm{d}A_s$ is $\int_0^t H_s \,\mathrm{d}\Pi_p^* A_s$.*

Exercise 4.4.12. *Let N be an $\mathcal{F}_t$ Poisson process and let T_n be its n'th event time. Let H be bounded and predictable. Show that $\frac{1}{t}\int_0^t H_s \,\mathrm{d}s$ converges in probability as t tends to infinity if and only if $\frac{1}{n}\sum_{k=1}^n H_{T_k}$ converges in probability as n tends to infinity, and in the affirmative, the limits agree.*

Exercise 4.4.13. *Let $M \in \mathcal{M}_\ell$ and let $A \in \mathcal{V}$ be predictable. Show that almost surely for all $t \geq 0$, $[M, A]_t = \sum_{0<s\leq t} \Delta M_s \Delta A_s$ and show that $[M, A] \in \mathcal{M}_\ell$.*

Exercise 4.4.14. *Let $A \in \mathcal{V}^i$ be predictable and let $M \in \mathcal{M}^b$. Show that the compensator of $\int_0^t M_s \,\mathrm{d}A_s$ is $\int_0^t M_{s-} \,\mathrm{d}A_s$.*

Exercise 4.4.15. *Let $X \in \mathcal{S}$ and let H be a predictable semimartingale. Show that almost surely, it holds for all $t \geq 0$ that $(\Delta H \cdot X)_t = \sum_{0<s\leq t} \Delta H_s \Delta X_s$.*

Exercise 4.4.16. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and let H be bounded, adapted and continuous. Show that for any fixed $t \geq 0$, $(W_{t+h} - W_t)^{-1} \int_t^{t+h} H_s \, \mathrm{d}W_s$ converges in probability to H_t as h tends to zero, where we define $\int_t^{t+h} H_s \, \mathrm{d}W_s = (H \cdot W)_{t+h} - (H \cdot W)_t$.*

Exercise 4.4.17. *Let X be a continuous process. Let $t \geq 0$ and let $t_k^n = kt2^{-n}$. Let $p > 0$ and assume that $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^p$ is convergent in probability. Show that $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^q$ converges to zero in probability for $q > p$.*

Exercise 4.4.18. *Let $0 < H < 1$ and X be a continuous adapted process such that X has finite-dimensional distributions which are normally distributed with mean zero and such that for any s and t with $s, t \geq 0$, $EX_sX_t = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. Such a process is called a fractional Brownian motion with Hurst parameter H. Show that if $H = \frac{1}{2}$, then X has the distribution of a Brownian motion. Show that if $H \neq \frac{1}{2}$, then X is not in $\mathbf{c}\mathcal{S}$.*

Exercise 4.4.19. *Let W be a p-dimensional $\mathcal{F}_t$ Brownian motion. Let $f : \mathbb{R}^p \to \mathbb{R}$ be C^2. Show that $f(W_t)$ is a continuous local martingale if $\sum_{i=1}^p \frac{\partial^2 f}{\partial x_i^2}(x) = 0$ for all $x \in \mathbb{R}^p$.*

Exercise 4.4.20. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^2. Show that $f(t, W_t)$ is a continuous local martingale if $\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0$ for $(t,x) \in \mathbb{R}^2$. Show that in the affirmative, it holds that $f(t, W_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s, W_s) \, \mathrm{d}W_s$.*

Exercise 4.4.21. *Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and let $f : \mathbb{R}_+ \to \mathbb{R}$ be continuous. Show that $f \in \mathfrak{I}$ in the sense that the process $(t, \omega) \mapsto f(t)$ is in $\mathfrak{I}$. Fix $t \geq 0$ and find the distribution of $\int_0^t f(s) \, \mathrm{d}W_s$.*

Exercise 4.4.22. *Let $X, Y \in \mathcal{S}$ and $f, g \in C^2(\mathbb{R})$. With $f(X)_t = f(X_t)$ and $g(Y)_t = g(Y_t)$, show that $[f(X), g(Y)]_t = \int_0^t f'(X_s)g'(Y_s) \, \mathrm{d}[X^c, Y^c]_s + \sum_{0<s\leq t} \Delta f(X_s)\Delta g(Y_s)$ for all $t \geq 0$ up to indistinguishability. Use this to identify the quadratic variation of W^p when W is an $\mathcal{F}_t$ Brownian motion and $p \geq 1$.*

Exercise 4.4.23. *Let W be a p-dimensional $\mathcal{F}_t$ Brownian motion. Find the quadratic co-variation process of $W_t^i W_t^j$ for $i, j \leq p$.*

Exercise 4.4.24. *Assume that $M \in \mathcal{M}^b$ and put $t_k^n = kt2^{-n}$. Show that the sequence $\sum_{k=1}^{2^n} (M_{t_k} - M_{t_{k-1}})^2$ converges in $\mathcal{L}^1$ to $[M]_t$.*

Chapter 5

Conclusion

In this final chapter, we review our results and give directions for further reading.

We begin by reviewing each chapter by turn and commenting on relevant litterature. As stated in the introduction, the theory covered here is intended as a short and rigorous path through the core of stochastic integration theory. The results of Chapter 1 are for the most part quite standard and can be found in several books on continuous-time stochastic processes, Karatzas & Shreve (1991) and Rogers & Williams (2000a) are a good sources in this regard.

Throughout this monograph, progressive measurability is invoked in order to apply Lemma 1.1.12. An interesting observation is that the apparently weaker requirement of being measurable and adapted implies the existence of a progressive modification, see for example Section IV.30 of Dellacherie & Meyer (1978) or Kaden & Potthoff (2005).

In Theorem 1.3.6, Lemma A.3.7 was applied to obtain the existence of the quadratic variation process for bounded martingales with initial value zero. The method of taking convex combinations to obtain convergence has been used several times previously in probability theory, see for example Lemma A1.1 of Delbaen & Schachermayer (1994) as well as the results in Delbaen & Schachermayer (1996) and Beiglböck et al. (2012).

The theme of Chapter 2 is that of the "general theory of processes", and is also covered in Dellacherie & Meyer (1978), He et al. (1992) and to some degree in Protter (2005). The main result of Section 2.1, Theorem 2.1.12, is a version of the PFA Theorem, see Theorem VI.12.6 of Rogers & Williams (2000b). In Section 2.2, the σ-algebra $\mathcal{F}_{T-}$ is introduced. For more on this, see also Chung & Doob (1965).

In Chapter 3, local martingales are introduced. Our proof in Section 3.2 of the existence of the compensator is based on ideas from Beiglböck et al. (2012). Other proofs may be obtained as corollaries of the Doob-Meyer decomposition, see for example Rogers & Williams (2000b) or Protter (2005) for this approach. Alternatively, an approach based on the predictable projection mapping may be applied, this approach generally requires the section theorems for uniqueness of the predictable projection. See He et al. (1992), Rogers & Williams (2000b) or Elliott (1982) for this approach.

The proof of the existence of the quadratic variation process in Section 3.3 is based on the decomposition given in the fundamental theorem of local martingales. Alternative approaches are given in for example Rogers & Williams (2000b) and He et al. (1992), based on considering the case of $\mathcal{M}^2$, in particular Rogers & Williams (2000b) proves a decomposition of $\mathcal{M}^2$ into continuous and purely discontinuous parts and proceeds from there. In Protter (2005) and Jacod & Shiryaev (2003), the stochastic integral is constructed before the quadratic variation, and the quadratic variation is then introduced as the remainder term in the integration-by-parts formula.

The construction of the stochastic integral with respect to a local martingale carried out in Theorem 4.2.8 of Chapter 4 is inspired by the proof given in Chapter IX of He et al. (1992). However, while He et al. (1992) applies a decomposition into continuous and purely discontinuous parts and a characterisation of the jump structure of purely discontinuous martingales based on the predictable projection, we apply the fundamental theorem of local martingales. In both cases, however, the Riesz representation theorem for $\mathcal{M}^2$ is invoked to obtain existence in some part of the existence proof. Alternative approaches are given in Rogers & Williams (2000b), Jacod & Shiryaev (2003) and Kallenberg (2002), based on first defining the integral for elementary types of processes and then extending by linearity and continuity requirements. An entirely different approach to the construction of the stochatic integral is given in Protter (2005), where the concept of a semimartingale is introduced as a process where the integral may be defined for left-continuous processes, afterwards proceeding to prove equivalence with the ordinary definition and finally extending the integral to predictable and locally bounded processes.

Our proof of Itô's formula in Section 4.3 is based on the methods used in Protter (2005). Another approach is given in Rogers & Williams (2000b) based on the integration-by-parts formula and approximation by polynomials. Elliott (1982) gives a proof based on successive approximations of semimartingales by more regular processes.

As regards further reading, Chapter IV of Rogers & Williams (2000b) includes some pointers for applications of the stochastic integral for continuous semimartingales. For topics covering the discontinuous case, He et al. (1992) contains several relevant chapters on for example changes of measure, martingale spaces, stochastic differential equations, martingale representation and weak convergence of semimartingales. Protter (2005) also covers many of the same topics, and also includes a chapter on expansion of filtrations. For a detailed account of many topics related to martingale theory, see Jacod (1979).

Appendix A

Appendices

In these appendices, we review the general analysis, measure theory and probability theory which are used in the main text, but whose subject matter is either taken to be more or less well-known or taken to be sufficiently different from our main interests to merit separation.

A.1 Measure theory and analysis

We first recall two fundamental results of basic measure theory.

Lemma A.1.1 (Dynkin's lemma)**.** *Let E be some set, and let $\mathbb{E}$ be a family of subsets of E which is stable under intersections. Let $\mathcal{H}$ be another family of subsets of E such that $E \in \mathcal{H}$, if $A, B \in \mathcal{H}$ with $A \subseteq B$ then $B \setminus A \in \mathcal{H}$ and if (A_n) is an increasing sequence in $\mathcal{H}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{H}$. Such a family is called a Dynkin class. If $\mathbb{E} \subseteq \mathcal{H}$, then $\sigma(\mathbb{E}) \subseteq \mathcal{H}$.*

Proof. See Theorem 2.1.3 of Karatzas & Shreve (1991). □

Theorem A.1.2. *Let $(E, \mathcal{E})$ be a measurable space. Let $\mathcal{A}$ be an algebra generating $\mathcal{E}$, and let $\mathcal{H}$ be such that whenever (H_n) is an increasing sequence in $\mathcal{H}$, then $\cup_{n=1}^{\infty} H_n \in \mathcal{H}$, and whenever (H_n) is a decreasing sequence in $\mathcal{H}$, then $\cap_{n=1}^{\infty} H_n \in \mathcal{H}$. We say that $\mathcal{H}$ is a monotone class. If $\mathcal{A} \subseteq \mathcal{H}$, then $\mathcal{E} \subseteq \mathcal{H}$ as well.*

Proof. See Theorem 1.3.9 of Ash (2000). $\square$

Next, we consider some results on signed measures. Let $(E, \mathcal{E})$ be a measurable space. A signed measure on $(E, \mathcal{E})$ is a mapping $\mu : \mathcal{E} \to \mathbb{R}$ such that $\mu(\emptyset) = 0$ and whenever (A_n) is a sequence of disjoint sets in $\mathcal{E}$, $\sum_{n=1}^\infty |\mu(A_n)|$ is convergent and $\mu(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$.

Theorem A.1.3. *Let μ be a signed measure on $(E, \mathcal{E})$. There exists a bounded nonnegative measure $|\mu|$ on $(E, \mathcal{E})$ such that $|\mu|(A) = \sup \sum_{n=1}^\infty |\mu(A_n)|$, where the supremum is taken over all mutually disjoint sequences (A_n) in $\mathcal{E}$ with $A = \cup_{n=1}^\infty A_n$. The nonnegative measure $|\mu|$ is called the total variation measure of μ. In particular, $|\mu(A)| \leq |\mu|(A) \leq |\mu|(E)$, so every signed measure is bounded.*

Proof. See Theorem 6.2 and Theorem 6.4 of Rudin (1987). $\square$

Lemma A.1.4. *Let μ and ν be two signed measures on a measurable space $(E, \mathcal{E})$. Let $\mathcal{E} = \sigma(\mathbb{E})$, where $\mathbb{E}$ is stable under intersections. If μ and ν are equal on $\mathbb{E}$ and satisfy $\mu(E) = \nu(E)$, they are equal on $\mathcal{E}$.*

Proof. We apply Lemma A.1.1. Let $\mathcal{H} = \{A \in \mathcal{E} | \mu(A) = \nu(A)\}$. We wish to show that $\mathcal{H}$ is a Dynkin class. By our assumptions, $\mu(E) = \nu(E)$, and so $E \in \mathcal{H}$. If $A, B \in \mathcal{H}$ with $A \subseteq B$, we obtain the equality $\mu(B \setminus A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B \setminus A)$, so $B \setminus A \in \mathcal{H}$. Finally, if (A_n) is an increasing sequence of sets in $\mathcal{H}$, we find that $\{A_1, A_2 \setminus A_1, A_3 \setminus A_2, \ldots\}$ is a sequence of disjoint sets in $\mathcal{H}$, and therefore,

$$\begin{aligned}
\mu(\cup_{n=1}^\infty A_n) &= \mu(A_1 \cup (\cup_{n=2}^\infty A_n \setminus A_{n-1})) = \mu(A_1) + \sum_{n=2}^\infty \mu(A_n \setminus A_{n-1}) \\
&= \nu(A_1) + \sum_{n=2}^\infty \nu(A_n \setminus A_{n-1}) = \nu(A_1 \cup (\cup_{n=2}^\infty A_n \setminus A_{n-1})) = \nu(\cup_{n=1}^\infty A_n),
\end{aligned}$$

where the sums converge absolutely by the definition of a signed measure. Thus, $\cup_{n=1}^\infty A_n \in \mathcal{H}$, and so $\mathcal{H}$ is a Dynkin class. Therefore, $\mathcal{H} = \sigma(\mathbb{E}) = \mathcal{E}$, and so $\mu = \nu$. $\square$

Theorem A.1.5 (Jordan-Hahn decomposition)**.** *Let μ be a signed measure on $(E, \mathcal{E})$. There is a unique pair of positive bounded singular measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$, given by $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu^- = \frac{1}{2}(|\mu| - \mu)$. This decomposition also satisfies $|\mu| = \mu^+ + \mu^-$. We call this the Jordan-Hahn decomposition of μ.*

Proof. By Section 6.6 of Rudin (1987) and Theorem 6.14 of Rudin (1987), the explicit construction of μ^+ and μ^- satisfies the requirements of the theorem. For uniqueness, assume that $\mu = \nu^+ - \nu^-$, where ν^+ and ν^- is another pair of singular positive bounded measures. Assume that ν^+ is concentrated on F^+ and ν^- is concentrated on F^-, while μ^+ is concentrated on E^+ and μ^- is concentrated on E^-, where F^+ and F^- are disjoint and E^+ and E^- are disjoint. For any $A \in \mathcal{E}$, we then have the inequalities $\mu(A\cap E^+\cap F^-) = \mu^+(A\cap F^-) \geq 0$ and $\mu(A \cap E^+ \cap F^-) = -\nu^-(A \cap E^+) \leq 0$, yielding $\mu(A \cap E^+ \cap F^-) = 0$, as well as the inequalities $\mu(A \cap E^- \cap F^+) = \nu^+(A \cap E^-) \geq 0$ and $\mu(A \cap E^- \cap F^+) = -\mu^-(A \cap F^+) \leq 0$, yielding $\mu(A \cap E^- \cap F^+) = 0$. Applying these results, we obtain

$$\begin{aligned}
\mu^+(A) &= \mu^+(A \cap E^+) = \mu(A \cap E^+) = \mu(A \cap E^+ \cap F^+) \\
&= \mu(A \cap F^+) = \nu^+(A \cap F^+) = \nu^+(A),
\end{aligned}$$

so μ^+ and ν^+ are equal, and therefore μ^- and ν^- are equal as well. □

Lemma A.1.6. *Let μ be a signed measure on $(E, \mathcal{E})$. Let $\mathcal{A}$ be an algebra generating $\mathcal{E}$. Then $|\mu|(E) = \sup \sum_{k=1}^n |\mu(A_k)|$, where the sum is taken over finite disjoint partitions (A_k) of A, and each element A_k is in $\mathcal{A}$.*

Proof. We first show that $|\mu|(E) = \sup \sum_{k=1}^n |\mu(A_k)|$, where the sum is taken over finite disjoint partitions (A_k) of E, and each element A_k is in $\mathcal{E}$. To this end, let $\varepsilon > 0$. There is a countable disjoint measurable partition (A_n) of E such that $|\mu|(E) \leq \varepsilon + \sum_{n=1}^\infty |\mu(A_n)|$. Since $|\mu|$ is a bounded positive measure, the sum $\sum_{n=1}^\infty |\mu|(A_n)$ is convergent, and therefore, there is k such that $|\mu(\cup_{n=k}^\infty A_n)| = |\sum_{n=k}^\infty \mu(A_n)| \leq \sum_{n=k}^\infty |\mu(A_n)| \leq \sum_{n=k}^\infty |\mu|(A_n) \leq \varepsilon$. As all the numbers in the chain of inequalities are nonnegative, we find in particular that $||\mu(\cup_{n=k}^\infty A_n)| - \sum_{n=k}^\infty |\mu(A_n)|| \leq \varepsilon$ and thus

$$|\mu|(E) \leq \varepsilon + \sum_{n=1}^\infty |\mu(A_n)| = \varepsilon + \sum_{n=1}^{k-1} |\mu(A_n)| + \sum_{n=k}^\infty |\mu(A_n)| \leq 2\varepsilon + |\mu(\cup_{n=k}^\infty A_n)| + \sum_{n=1}^{k-1} |\mu(A_n)|,$$

and since the family of sets $A_1, \ldots, A_{k-1}, \cup_{n=k}^\infty A_n$ is a finite disjoint partition of E with each element in $\mathcal{E}$, and $\varepsilon > 0$ was arbitrary, we conclude that $|\mu|(E) = \sup \sum_{k=1}^n |\mu(A_k)|$, where the supremum is over finite disjoint measurable partitions of E.

Next, we show that it suffices to consider partitions with each element in $\mathcal{A}$. Let $\varepsilon > 0$, we need to locate a finite disjoint partition (A_n) of E with elements from $\mathcal{A}$ such that $|\mu|(E) \leq \varepsilon + \sum_{n=1}^k |\mu(A_n)|$. From what we have just shown, there is a finite disjoint partition (A_n) of E with each A_n in $\mathcal{E}$ such that $|\mu|(E) \leq \varepsilon + \sum_{n=1}^k |\mu(A_n)|$. For any $n \leq k$, we may use Theorem 1.3.11 of Ash (2000) to obtain some $B_n \in \mathcal{A}$ with $|\mu|(A_n \Delta B_n) \leq \frac{1}{k}\varepsilon 2^{-k}$, where the symmetric difference $A_n \Delta B_n$ is defined by $A_n \Delta B_n = (A_n \cap B_n^c) \cup (A_n^c \cap B_n)$.

Now let $\mathbb{P}_k$ denote the set of all subsets of $\{1, \ldots, k\}$, and define the set C_α for any $\alpha \in \mathbb{P}_k$ by putting $C_\alpha = \{x \in E \mid \forall\ n \leq k : x \in B_n \text{ if } n \in \alpha \text{ and } x \in B_n^c \text{ if } n \notin \alpha\}$. C_α is the intersection of the B_n's with $n \in \alpha$ and the B_n^c's with $n \notin \alpha$. In particular, the family $(C_\alpha)_{\alpha \in \mathbb{P}_k}$ consists of mutually disjoint sets. As for each $x \in E$ and each $n \leq k$, we either have $x \in B_n$ or $x \in B_n^c$, the family $(C_\alpha)_{\alpha \in \mathbb{P}_k}$ is a finite disjoint partition of E, and as $\mathcal{A}$ is an algebra, $C_\alpha \in \mathcal{A}$ for all $\alpha \in \mathbb{P}_k$. We claim that $|\mu|(E) \leq 3\varepsilon + \sum_{\alpha \in \mathbb{P}_k} |\mu(C_\alpha)|$.

To prove this, we first note that for any $n \leq k$, we have

$$\begin{aligned}
||\mu(A_n)| - |\mu(B_n)|| &\leq |\mu(A_n) - \mu(B_n)| \\
&\leq |\mu(A_n \cap B_n) + \mu(A_n \cap B_n^c) - (\mu(A_n \cap B_n) + \mu(A_n^c \cap B_n))| \\
&= |\mu(A_n \cap B_n^c) - \mu(A_n^c \cap B_n)| \leq |\mu(A_n \cap B_n^c)| + |\mu(A_n^c \cap B_n)| \\
&\leq |\mu|(A_n \cap B_n^c) + |\mu|(A_n^c \cap B_n) = |\mu|(A_n \Delta B_n),
\end{aligned}$$

which is less than $\frac{1}{k}\varepsilon 2^{-k}$. Therefore, we obtain

$$|\mu|(E) \leq \varepsilon + \sum_{n=1}^{k} |\mu(A_n)| \leq \varepsilon + \sum_{n=1}^{k} |\mu(B_n)| + \sum_{n=1}^{k} ||\mu(A_n)| - |\mu(B_n)|| \leq 2\varepsilon + \sum_{n=1}^{k} |\mu(B_n)|.$$

Note that $B_n = \cup_{\alpha : n \in \alpha} C_\alpha$, with each pair of sets in the union being mutually disjoint. We will argue that $|\mu|(B_n) \leq |\mu|(C_{\{n\}}) + \frac{1}{k}\varepsilon$. To see this, consider some $\alpha \in \mathbb{P}_n$ with more than one element, assume for definiteness that $n, m \in \alpha$ with $n \neq m$. As the A_n's are disjoint, we then find

$$\begin{aligned}
|\mu|(C_\alpha) &\leq |\mu|(B_n \cap B_m) = |\mu|(B_n \cap A_n \cap B_m) + |\mu|(B_n \cap A_n^c \cap B_m) \\
&\leq |\mu|(A_n \cap B_m) + |\mu|(B_n \cap A_n^c) = |\mu|(A_n \cap A_m^c \cap B_m) + |\mu|(B_n \cap A_n^c) \\
&\leq |\mu|(A_m^c \cap B_m) + |\mu|(B_n \cap A_n^c) \leq |\mu|(A_m \Delta B_m) + |\mu|(A_n \Delta B_n),
\end{aligned}$$

which is less than $\frac{2}{k}\varepsilon 2^{-k}$. We now note, using $C_{\{n\}} \subseteq B_n$, that

$$\begin{aligned}
|\mu(B_n)| &\leq |\mu(C_{\{n\}})| + |\mu(B_n) - \mu(C_{\{n\}})| \\
&= |\mu(C_{\{n\}})| + |\mu(B_n \setminus C_{\{n\}})| \\
&\leq |\mu(C_{\{n\}})| + |\mu|(B_n \setminus C_{\{n\}}).
\end{aligned}$$

However, $|\mu|(B_n \setminus C_{\{n\}}) = |\mu|(B_n) - |\mu|(C_{\{n\}}) = \sum_{n \in \alpha, \alpha \neq \{n\}} |\mu|(C_\alpha)$ and as there are less than 2^{k-1} elements in the sum, with each element according to what was alerady proven has a value of less than $\frac{2}{k}\varepsilon 2^{-k}$, we find $|\mu|(B_n) \leq |\mu|(C_{\{n\}}) + \frac{1}{k}\varepsilon$. We may now conclude $|\mu|(E) \leq 2\varepsilon + \sum_{n=1}^{k} |\mu(B_n)| \leq 3\varepsilon + \sum_{n=1}^{k} |\mu(C_{\{n\}})| \leq 3\varepsilon + \sum_{\alpha \in \mathbb{P}_n} |\mu(C_\alpha)|$. As ε was arbitrary, we conclude that $|\mu(E)| = \sup \sum_{n=1}^{k} |\mu(A_n)|$, where the sum is taken over finite disjoint partitions (A_k) of E, and each element A_k is in $\mathcal{A}$. $\square$

Next, we consider two results on algebras and σ-algebras.

Lemma A.1.7. *Let E be some set, and let $\mathbb{A}$ be a set family on E. Let $\mathcal{A}$ denote the family of finite unions of elements of $\mathbb{A}$. Assume that*

1. *$\mathbb{A}$ contains E.*
2. *$\mathbb{A}$ is stable under finite intersections.*
3. *The complements of sets in $\mathbb{A}$ are in $\mathcal{A}$.*

Then $\mathcal{A}$ is an algebra.

Proof. We need to prove that $\mathcal{A}$ is stable under finite unions and intersections. The family $\mathcal{A}$ is stable under finite unions by construction. It will therefore suffice to show stability under finite intersections. As we have $A \cap B = (A^c \cup B^c)^c$, it will be sufficient to show stability under complements. Let $A \in \mathcal{A}$ with $A = \cup_{k=1}^n A_k$, where $A_k \in \mathbb{A}$ for $k \leq n$. By assumption, $A_k^c \in \mathcal{A}$, so $A_k^c = \cup_{i=1}^{m_k} B_{ik}$ for some $B_{ik} \in \mathbb{A}$. We then have

$$A^c = \cap_{k=1}^n A_k^c = \cap_{k=1}^n \cup_{i=1}^{m_k} B_{ik} = \cup_{i_1=1}^{m_1} \cdots \cup_{i_n=1}^{m_n} \cap_{k=1}^n B_{i_k k}.$$

Since $\mathbb{A}$ is stable under finite intersections, we obtain $A^c \in \mathcal{A}$, as desired. $\square$

Lemma A.1.8. *Let $(E, \mathcal{E})$ be a measurable space endowed with a bounded positive measure μ, and let $\mathcal{A}$ be an algebra generating $\mathcal{E}$. For any $A \in \mathcal{E}$, it holds that*

$$\mu(A) = \sup\{\mu(B) | B \in \mathcal{A}_\delta, B \subseteq A\} = \inf\{\mu(B) | B \in \mathcal{A}_\sigma, A \subseteq B\},$$

where $\mathcal{A}_\delta$ and $\mathcal{A}_\sigma$ denotes the family of countable intersections and countable unions of elements in $\mathcal{A}$, respectively.

Proof. Let $\mathcal{H}$ be the family of sets A in $\mathcal{E}$ with the desired approximation property. We wish to show that $\mathcal{H}$ is equal to $\mathcal{E}$. Clearly, $\mathcal{H}$ contains the algebra $\mathcal{A}$ generating $\mathcal{E}$. Therefore, by Theorem A.1.2, we may conclude that $\mathcal{H}$ is equal to $\mathcal{E}$ if only we can prove that $\mathcal{H}$ is a monotone class.

To this end, we first let (A_n) be an increasing sequence of elements in $\mathcal{H}$ and we define $A = \cup_{n=1}^\infty A_n$. We wish to show that $A \in \mathcal{H}$ as well. We first consider the approximation of A by elements of $\mathcal{A}_\delta$. Fix $\varepsilon > 0$. We may use the continuity of μ to find n so large that $\mu(A \setminus A_n) \leq \frac{\varepsilon}{2}$. Let $B \in \mathcal{A}_\delta$ with $B \subseteq A_n$ be such that $\mu(A_n) \leq \mu(B) + \frac{\varepsilon}{2}$, we then obtain $\mu(A) = \mu(A \setminus A_n) + \mu(A_n) \leq \mu(B) + \varepsilon$. As $B \subseteq A_n \subseteq A$, we obtain the inequality $\mu(A) \leq \sup\{\mu(B)|B \in \mathcal{A}_\delta, B \subseteq A\}$. As the other inequality always holds, this leads os to conclude that $\mu(A) = \sup\{\mu(B)|B \in \mathcal{A}_\delta, B \subseteq A\}$, showing that A may be approximated from the inside by elements of $\mathcal{A}_\delta$. Next, we consider the approximation of A from the outside by elements of $\mathcal{A}_\sigma$. For each n, take $B_n \in \mathcal{A}_\sigma$ such that $A_n \subseteq B_n$ and $\mu(B_n \setminus A_n) \leq \varepsilon 2^{-n}$. Define $B = \cup_{n=1}^\infty B_n$, then $B \in \mathcal{A}_\sigma$ as well. Furthermore, we have $A = \cup_{n=1}^\infty A_n \subseteq \cup_{n=1}^\infty B_n = B$, and we find

$$\mu(B) = \lim_n \mu(\cup_{k=1}^n B_k) = \lim_n \mu(\cup_{k=1}^n B_k \setminus A_n) + \mu(A_n) = \mu(A) + \lim_n \mu(\cup_{k=1}^n B_k \setminus A_n).$$

Here, as (A_n) is increasing, (A_n^c) is decreasing, and so we have

$$\mu(\cup_{k=1}^n B_k \setminus A_n) \leq \sum_{k=1}^n P(B_k \cap A_n^c) \leq \sum_{k=1}^n P(B_k \cap A_k^c) \leq \sum_{k=1}^n \varepsilon 2^{-k} \leq \varepsilon,$$

allowing us to conclude that $\mu(B) \leq \mu(A) + \varepsilon$, and so $\mu(A) \geq \inf\{\mu(B)|B \in \mathcal{A}_\sigma, A \subseteq B\}$. And as the other inequality is obvious, we find that we have equality. This finally allows us to conclude that $A \in \mathcal{H}$.

We have now shown that when (A_n) is an increasing sequence of sets in $\mathcal{H}$, then $\cup_{n=1}^\infty A_n \in \mathcal{H}$ as well. To prove the analogous result for decreasing sequences, we first show that $\mathcal{H}$ is stable under complements. To this end, first note that if (B_n) is a sequence in $\mathcal{A}$, then $(\cup_{n=1}^\infty B_n)^c = \cap_{n=1}^\infty B_n^c$, where (B_n^c) is also a sequence in $\mathcal{A}$. Therefore, $B \in \mathcal{A}_\delta$ if and only if $B^c \in \mathcal{A}_\sigma$. Now let $A \in \mathcal{H}$. We then find that

$$\begin{aligned}
\mu(A^c) &= \mu(E) - \mu(A) \\
&= \mu(E) - \sup\{\mu(B)|B \in \mathcal{A}_\delta, B \subseteq A\} \\
&= \mu(E) + \inf\{-\mu(B)|B \in \mathcal{A}_\delta, B \subseteq A\} \\
&= \inf\{\mu(B^c)|B \in \mathcal{A}_\delta, B \subseteq A\} \\
&= \inf\{\mu(B)|B \in \mathcal{A}_\sigma, A^c \subseteq B\},
\end{aligned}$$

and in the same manner, $\mu(A^c) = \sup\{\mu(B)|B \in \mathcal{A}_\delta, B \subseteq A^c\}$. Thus $A^c \in \mathcal{H}$. In order to complete the proof that $\mathcal{H}$ is a monotone class, now assume that (A_n) is a decreasing sequence in $\mathcal{H}$. Then (A_n^c) is an increasing sequence in $\mathcal{H}$, so from what we already have shown, $\cup_{n=1}^\infty A_n^c \in \mathcal{H}$. By stability under complements, this yields $\cap_{n=1}^\infty A_n \in \mathcal{H}$ as well. Theorem A.1.2 now allows us to conclude that $\mathcal{H}$ is equal to $\mathcal{E}$ and so the lemma holds. □

Finally, we introduce integration measures and prove versions of the Tonelli and Fubini theorems. For these purposes, the following lemma will come in handy. We use the notation that for set families $\mathbb{E}$ and $\mathbb{F}$ of subsets of sets E and F, respectively, $\mathbb{E}\times\mathbb{F}$ denotes the family $\{A \times B \mid A \in \mathbb{E}, B \in \mathbb{F}\}$.

Lemma A.1.9. *Let $(E,\mathcal{E})$ and $(F,\mathcal{F})$ be two measurable spaces. Let $\mathbb{E}$ be a generator for $\mathcal{E}$ and let $\mathbb{F}$ be a generator for $\mathcal{F}$. Assume that there are sequences $(E_n) \subseteq \mathbb{E}$ and $(F_n) \subseteq \mathbb{F}$ such that $E = \cup_{n=1}^\infty E_n$ and $F = \cup_{n=1}^\infty F_n$. It then holds that $\mathcal{E}\otimes\mathcal{F} = \sigma(\mathbb{E}\times\mathbb{F})$.*

Proof. First define $\overline{\mathbb{E}} = \mathbb{E}\cup\{E\}$ and $\overline{\mathbb{F}} = \mathbb{F}\cup\{F\}$. We begin by proving $\sigma(\mathbb{E}\times\mathbb{F}) = \sigma(\overline{\mathbb{E}}\times\overline{\mathbb{F}})$. The inclusion $\sigma(\mathbb{E}\times\mathbb{F}) \subseteq \sigma(\overline{\mathbb{E}}\times\overline{\mathbb{F}})$ is immediate, so it will suffice to show $\sigma(\overline{\mathbb{E}}\times\overline{\mathbb{F}}) \subseteq \sigma(\mathbb{E}\times\mathbb{F})$, and to do so, it will suffice to prove $\overline{\mathbb{E}}\times\overline{\mathbb{F}} \subseteq \sigma(\mathbb{E}\times\mathbb{F})$. To this end, let $A \in \mathbb{E}$. We then have $A\times F_n \in \mathbb{E}\times\mathbb{F}$, so $A\times F = \cup_{n=1}^\infty A\times F_n \in \sigma(\mathbb{E}\times\mathbb{F})$. Thus, all sets of the form $A\times F$, where $A \in \mathbb{E}$, is in $\sigma(\mathbb{E}\times\mathbb{F})$. In the same manner, we may argue that all sets of the form $E\times B$, where $B \in \mathbb{F}$, is in $\sigma(\mathbb{E}\times\mathbb{F})$. Finally, we also have $E\times F = \cup_{n=1}^\infty E_n\times F_n \in \sigma(\mathbb{E}\times\mathbb{F})$. Thus, $\overline{\mathbb{E}}\times\overline{\mathbb{F}} \subseteq \sigma(\mathbb{E}\times\mathbb{F})$, as claimed, yielding $\sigma(\mathbb{E}\times\mathbb{F}) = \sigma(\overline{\mathbb{E}}\times\overline{\mathbb{F}})$.

Now, let $\mathcal{H}$ be the σ-algebra generated by $\overline{\mathbb{E}}\times\overline{\mathbb{F}}$. From what we have just shown, to prove the claim of the lemma, it will suffice to prove $\mathcal{E}\otimes\mathcal{F} = \mathcal{H}$. It is immediate that $\mathcal{H} \subseteq \mathcal{E}\otimes\mathcal{F}$, we need to prove the opposite inclusion. To do so, it will suffice to prove $\mathcal{E}\times\mathcal{F} \subseteq \mathcal{H}$. In order to obtain this, let $\mathbb{F}'$ be the family of sets $B \in \mathcal{F}$ such that $E\times B \in \mathcal{H}$. Then $\mathbb{F}'$ is stable under complements and countable unions, and $\overline{\mathbb{F}} \subseteq \mathbb{F}'$. In particular, $F \in \mathbb{F}'$. We conclude that $\mathbb{F}'$ is a σ-algebra containing $\overline{\mathbb{F}}$, therefore $\mathcal{F} \subseteq \mathbb{F}'$. This shows that $E\times B \in \mathcal{H}$ for any $F \in \mathcal{F}$. Analogously, we can prove that $A\times F \in \mathcal{H}$ for any $A \in \mathcal{E}$. Letting $A \in \mathcal{E}$ and $B \in \mathcal{F}$, we then obtain $A\times B = (A\times F)\cap(E\times B) \in \mathcal{H}$, as desired. We conclude $\mathcal{E}\times\mathcal{F} \subseteq \mathcal{H}$ and therefore $\mathcal{E}\otimes\mathcal{F} \subseteq \mathcal{H}$, as was to be proved. □

Next, we consider the existence and properties of integration measures. We will restrict our attention to the case where $(E,\mathcal{E})$ is countably generated in the sense that there exists a countable generating family for $\mathcal{E}$. We begin by considering a few lemmas.

Lemma A.1.10. *Assume that $(E,\mathcal{E})$ is a countably generated measure space. Then, there exists a sequence of finite partitions $(\mathcal{P}_n)_{n\geq 1}$ of E with $\mathcal{P}_n \subseteq \mathcal{E}$ such that $(\sigma(\mathcal{P}_n))_{n\geq 1}$ is increasing and such that $\mathcal{E}$ is generated by $\cup_{n=1}^\infty \mathcal{P}_n$.*

Proof. This is shown on p. 209 of Stroock (2010). □

Lemma A.1.11. *Assume that $(E,\mathcal{E})$ is a countably generated measure space. Then, there exists a countable algebra generating $\mathcal{E}$.*

Proof. By Lemma A.1.10, there exists a sequence of finite partitions $(\mathcal{P}_n)_{n\geq 1}$ of E with $\mathcal{P}_n \subseteq \mathcal{E}$ such that $(\sigma(\mathcal{P}_n))_{n\geq 1}$ is increasing and such that $\mathcal{E}$ is generated by $\cup_{n=1}^{\infty}\mathcal{P}_n$. Then $\mathcal{E}$ is also generated by $\cup_{n=1}^{\infty}\sigma(\mathcal{P}_n)$. As $\mathcal{P}_n$ is finite, $\sigma(\mathcal{P}_n)$ is finite as well. Therefore, $\cup_{n=1}^{\infty}\sigma(\mathcal{P}_n)$ is countable, and it is immediate that it is an algebra. □

Lemma A.1.12. *Let (ν_ω) be a family of signed measures on $(E,\mathcal{E})$. Assume that $\omega \mapsto \nu_\omega(A)$ is $\mathcal{F}$ measurable for all $A \in \mathcal{E}$. Then $\omega \mapsto |\nu_\omega|(A)$ is also $\mathcal{F}$ measurable for all $A \in \mathcal{E}$.*

Proof. Note that as $\mathcal{E}$ is countably generated, Lemma A.1.11 shows that $\mathcal{E}$ is generated by a countable algebra. Thus, we are in the setting of Section 2 of Dubins & Freedman (1964). Therefore, Theorem 2.9 of Dubins & Freedman (1964) yields the result. □

In the following, assume that $(E,\mathcal{E})$ is countably generated. Assume further given a measurable space $(\Omega,\mathcal{F})$ endowed with a probability measure P, and let (ν_ω) be a family of signed measures on $(E,\mathcal{E})$. Assume that $\omega \mapsto \nu_\omega(A)$ is $\mathcal{F}$ measurable for all $A \in \mathcal{E}$ and assume that $\int_\Omega |\nu_\omega|(E)\,\mathrm{d}P(\omega)$ is finite. We then say that (ν_ω) is a P-integrable $\mathcal{F}$ kernel on $\mathcal{E}$. Note that $\int_\Omega |\nu_\omega|(E)\,\mathrm{d}P(\omega)$ always is well-defined by Lemma A.1.12.

Theorem A.1.13. *Let (ν_ω) be a P-integrable $\mathcal{F}$-kernel on $\mathcal{E}$. There exists a unique signed measure λ on $\mathcal{F}\otimes\mathcal{E}$, called the integration of (ν_ω) with respect to P, uniquely characterized by the requirement that for $F \in \mathcal{F}$ and $A \in \mathcal{E}$, $\lambda(F\times A) = \int_F \nu_\omega(A)\,\mathrm{d}P(\omega)$. If each ν_ω is nonnegative, λ is nonnegative.*

Proof. Note that the proposed expression for $\lambda(F\times A)$ is well-defined, as $\omega \mapsto \nu_\omega(A)$ is $\mathcal{F}$ measurable for all $A \in \mathcal{E}$. Defining $\nu_\omega^+ = \frac{1}{2}(|\nu_\omega| + \nu_\omega)$ and $\nu_\omega^- = \frac{1}{2}(|\nu_\omega| - \nu_\omega)$, we know from Theorem A.1.5 that $\nu_\omega = \nu_\omega^+ - \nu_\omega^-$ is the Jordan-Hahn decomposition of ν_ω. By Lemma A.1.12, we find that $\omega \mapsto \nu_\omega^+(A)$ and $\omega \mapsto \nu_\omega^-(A)$ are $\mathcal{F}$ measurable for all $A \in \mathcal{E}$. By Theorem 4.20 of Pollard (2002), there exists two positive measures λ^+ and λ^- on $\mathcal{F}\otimes\mathcal{E}$ such that for any $F \in \mathcal{F}$ and $A \in \mathcal{E}$, $\lambda^+(F\times A) = \int_F \nu_\omega^+(A)\,\mathrm{d}P(\omega)$ and $\lambda^-(F\times A) = \int_F \nu_\omega^-(A)\,\mathrm{d}P(\omega)$. As $\int_\Omega |\nu_\omega|(E)\,\mathrm{d}P(\omega)$ is finite, both λ^+ and λ^- are bounded. Therefore, we may define $\lambda = \lambda^+ - \lambda^-$ and obtain a signed measure on $\mathcal{F}\otimes\mathcal{E}$ with the desired qualities. Clearly, if each ν_ω is nonnegative, ν_ω^- is zero for all ω and so λ^- is zero, such that λ is nonnegative in this case.

Uniqueness follows from Lemma A.1.4, since the class of sets $F\times A$ where $F \in \mathcal{F}$ and $A \in \mathcal{E}$ forms a generating family for $\mathcal{F}\otimes\mathcal{E}$ which is stable under intersections. □

In order to obtain Tonelli's and Fubini's theorems for integration measures, we first need to identify the total variation measure of an integration measure.

Lemma A.1.14. *Let (ν_ω) be a P-integrable $\mathcal{F}$-kernel on $\mathcal{E}$. There exists an $\mathcal{F}\otimes\mathcal{E}$-$\mathcal{B}$ measurable mapping $h:\Omega\times E\to\mathbb{R}$ taking its values in $\{-1,1\}$ such that for all $\omega\in\Omega$, the measure ν_ω^+ is concentrated on $\{x\in E \mid h(\omega,x)=1\}$ and the measure ν_ω^- is concentrated on $\{x\in E\mid h(\omega,x)=-1\}$.*

Proof. By Theorem 6.12 of Rudin (1987), for each $\omega\in\Omega$, there exists a $\mathcal{E}$-$\mathcal{B}$ measurable mapping $k(\omega,\cdot):E\to\mathbb{R}$ with values in $\{-1,1\}$ which is a version of the Radon-Nikodym derivative of ν_ω with respect to $|\nu_\omega|$. Now let $(\mathcal{P}_n)_{n\geq 1}$ be a sequence of finite partitions as given in Lemma A.1.10, and define

$$h_n(\omega,x)=\sum_{A\in\mathcal{P}_n}1_A(x)\frac{\nu_\omega(A)}{|\nu_\omega|(A)}1_{(|\nu_\omega|(A)>0)}.$$

The mapping h_n is $\mathcal{F}\otimes\mathcal{E}$-$\mathcal{B}$ measurable. Furthermore, define $G=\{\omega\in\Omega\mid |\nu_\omega|(E)\neq 0\}$. By Lemma A.1.12, G is $\mathcal{F}$ measurable. For $\omega\in G$, let $\mu_\omega=|\nu_\omega|/|\nu_\omega|(E)$, we then also have

$$h_n(\omega,x)=\sum_{A\in\mathcal{P}_n}1_A(x)\frac{1}{\mu_\omega(A)}\int k(\omega,x)\,\mathrm{d}\mu_\omega(x)1_{(\mu_\omega(A)>0)}.$$

From this, we see that $h_n(\omega,\cdot)$ is the conditional expectation of $k(\omega,\cdot)$ with respect to $\sigma(\mathcal{P}_n)$ on the probability space $(E,\mathcal{E},\mu_\omega)$. By arguments as in Theorem 5.2.7 of Stroock (2010), h_n then converges μ_ω almost surely to the conditional expectation of $k(\omega,\cdot)$ with respect to $\sigma(\cup_{n=1}^\infty\sigma(\mathcal{P}_n))$, which is almost surely equal to $k(\omega,\cdot)$.

Now define a mapping $h:\Omega\times E\to\mathbb{R}$ by letting $h(\omega,x)$ be the limit of h_n whenever this exists and is equal to either -1 or 1 and $\omega\in G$, and 1 otherwise. Then h is $\mathcal{F}\otimes\mathcal{E}$-$\mathcal{B}$ measurable and takes its values in $\{-1,1\}$. Fix $\omega\in\Omega$, we need to show that ν_ω^+ is concentrated on $\{x\in E\mid h(\omega,x)=1\}$ and ν_ω^- is concentrated on $\{x\in E\mid h(\omega,x)=-1\}$. If $\omega\in G^c$, it holds that ν_ω is zero, so the result trivially holds in this case. Consider instead $\omega\in G$. As $h_n(\omega,\cdot)$ converges μ_ω almost surely to $k(\omega,\cdot)$ in this case and $k(\omega,\cdot)$ takes its values in $\{-1,1\}$, it holds in particular that μ_ω almost surely, $h(\omega,\cdot)=k(\omega,\cdot)$. Therefore, this also holds $|\nu_\omega|$ almost surely, and so we obtain

$$\begin{aligned}\nu_\omega^+(\{x\in E\mid h(\omega,x)=-1\}) &= \nu_\omega^+(\{x\in E\mid k(\omega,x)=-1\})\\ &= \int 1_{\{x\in E\mid k(\omega,x)=-1\}}(y)k(\omega,y)\,\mathrm{d}|\nu_\omega|(y)\leq 0,\end{aligned}$$

so $\nu_\omega^+(\{x\in E\mid h(\omega,x)=-1\})=0$ and thus ν_ω^+ is concentrated on $\{x\in E\mid h(\omega,x)=1\}$. Similarly, we may obtain that ν_ω^- is concentrated on $\{x\in E\mid h(\omega,x)=-1\}$. This concludes the proof. $\square$

Lemma A.1.15. *Let (ν_ω) be a P-integrable $\mathcal{F}$-kernel on $\mathcal{E}$. Let λ be the integration of (ν_ω) with respect to P. The variation measure of λ is the integration of the P-integrable $\mathcal{F}$-kernel $(|\nu_\omega|)$ on $\mathcal{E}$.*

Proof. By Lemma A.1.14, there exists an $\mathcal{F}\otimes\mathcal{E}$-$\mathcal{B}$ measurable mapping $h : \Omega\times E \to \mathbb{R}$ taking its values in $\{-1, 1\}$ such that for all $\omega \in \Omega$, ν^+_ω is concentrated on $\{x \in E \mid h(\omega, x) = 1\}$ and ν^-_ω is concentrated on $\{x \in E \mid h(\omega, x) = -1\}$. Using Theorem A.1.13, let λ^+ and λ^- be the integrations of (ν^+_ω) and (ν^-_ω), respectively. By Lemma A.1.4, we obtain $\lambda = \lambda^+ - \lambda^-$. Applying Theorem 4.20 of Pollard (2002), we obtain

$$\begin{aligned}\lambda^+(\{(\omega, x) \in \Omega \times E \mid h(\omega, x) = -1\}) &= \int\int 1_{\{(\omega,x)\in\Omega\times E|h(\omega,x)=-1\}}\,\mathrm{d}\nu^+_\omega(x)\,\mathrm{d}P(\omega)\\ &= \int \nu^+_\omega(\{(\omega, x) \in \Omega \times E \mid h(\omega, x) = -1\})\,\mathrm{d}P(\omega),\end{aligned}$$

which is zero, so λ^+ is concentrated on $\{(\omega, x) \in \Omega \times E \mid h(\omega, x) = 1\}$. Similarly, λ^- is concentrated on $\{(\omega, x) \in \Omega \times E \mid h(\omega, x) = -1\}$. Therefore, we find that λ^+ and λ^- are singular. By the uniqueness statement of Theorem A.1.5, we conclude that $\lambda = \lambda^+ - \lambda^-$ is the Jordan-Hahn decomposition of λ, and thus $|\lambda| = \lambda^+ + \lambda^-$, also by Theorem A.1.5. As λ^+ and λ^- are the integrations of (ν^+_ω) and (ν^-_ω), respectively, this shows that $|\lambda|$ is the integration of $(|\nu_\omega|)$, as desired. □

Theorem A.1.16 (Tonelli's theorem for integration measures)**.** *Let P be a probability measure on $(\Omega, \mathcal{F})$, let $(E, \mathcal{E})$ be a measurable space and let (ν_ω) be a P-integrable $\mathcal{F}$-kernel on $\mathcal{E}$. Let λ be the integration of (ν_ω) with respect to P. For any nonnegative $\mathcal{F}\otimes\mathcal{E}$ measurable function $f : \Omega \times E \to [0,\infty]$, the following holds:*

1. *The mapping $x \mapsto f(\omega, x)$ is $\mathcal{E}$ measurable for each $\omega \in \Omega$.*
2. *The mapping $\omega \mapsto \int f(\omega, x)\,\mathrm{d}|\nu_\omega|(x)$ is $\mathcal{F}$ measurable.*
3. *$\int f(\omega, x)\,\mathrm{d}|\lambda|(\omega, x) = \int\int f(\omega, x)\,\mathrm{d}|\nu_\omega|(x)\,\mathrm{d}P(\omega)$.*

Proof. By Lemma A.1.15, $|\lambda|$ is the integration of the P-integrable $\mathcal{F}$-kernel $(|\nu_\omega|)_{\omega\in\Omega}$ on $\mathcal{E}$. Therefore, the result follows from Theorem 4.20 of Pollard (2002). □

Theorem A.1.17 (Fubini's theorem for integration measures)**.** *Let P be a probability measure on $(\Omega, \mathcal{F})$, let $(E, \mathcal{E})$ be a measurable space and let (ν_ω) be a P-integrable $\mathcal{F}$-kernel on $\mathcal{E}$. Let λ be the integration of (ν_ω) with respect to P. Let $f : \Omega\times E \to \mathbb{R}$ be an $\mathcal{F}\otimes\mathcal{E}$ measurable function which is integrable with respect to $|\lambda|$.*

1. *The mapping $x \mapsto f(\omega, x)$ is $\mathcal{E}$ measurable for each $\omega \in \Omega$.*

2. *For P almost all ω, the mapping $x \mapsto f(\omega, x)$ is integrable with respect to ν_ω, and the mapping $\omega \mapsto \int f(\omega, x)\,\mathrm{d}\nu_\omega(x)$ is $\mathcal{F}$ measurable and P integrable when put to zero whenever undefined.*

3. *It holds that $\int f(\omega, x)\,\mathrm{d}\lambda(\omega, x) = \int\int f(\omega, x)\,\mathrm{d}\nu_\omega(x)\,\mathrm{d}P(\omega)$.*

Proof. For the first claim, see Theorem 7.5 of Rudin (1987). As regards the second claim, Theorem A.1.16 yields that for each ω, $x \mapsto |f(\omega, x)|$ is $\mathcal{E}$ measurable, $\omega \mapsto \int |f(\omega, x)|\,|\,\mathrm{d}\nu_\omega|(x)$ is $\mathcal{F}$ measurable and $\int\int |f(\omega, x)|\,|\,\mathrm{d}\nu_\omega(x)|\,\mathrm{d}P(\omega) = \int |f(\omega, x)|\,|\,\mathrm{d}\lambda|(\omega, x)$, and the latter is finite by assumption. Therefore, $\int |f(\omega, x)|\,|\,\mathrm{d}\nu_\omega(x)|$ is finite for P almost all ω. In particular, for such ω, $x \mapsto f(\omega, x)$ is integrable with respect to ν_ω. Now let N be the null set such that $\int |f(\omega, x)|\,|\,\mathrm{d}\nu_\omega(x)|$ is infinite. As $\omega \mapsto \int |f(\omega, x)|\,|\,\mathrm{d}\nu_\omega|(x)$ is $\mathcal{F}$ measurable by Theorem A.1.16, we obtain $N \in \mathcal{F}$. Now let $f^+(\omega, x) = \max\{f(\omega, x), 0\}$ and $f^-(\omega, x) = \min\{f(\omega, x), 0\}$, and let $\lambda^+ = \frac{1}{2}(|\lambda| + \lambda)$ and $\lambda^- = \frac{1}{2}(|\lambda| - \lambda)$, By Theorem A.1.5, λ^+ and λ^- are then the positive and negative parts of λ in the Jordan-Hahn decomposition of λ. Applying Lemma A.1.15, we find that λ^+ is the integration of $(\nu_\omega^+)_{\omega\in\Omega}$ with respect to P, and λ^- is the integration of $(\nu_\omega^-)_{\omega\in\Omega}$ with respect to P. We then have

$$
\begin{aligned}
1_{N^c}(\omega)\int f(\omega, x)\,\mathrm{d}\nu_\omega(x) &= 1_{N^c}(\omega)\int f^+(\omega, x)\,\mathrm{d}\nu_\omega^+(x) - 1_{N^c}(\omega)\int f^-(\omega, x) df\nu_\omega^+(x) \\
&+ 1_{N^c}(\omega)\int f^+(\omega, x)\,\mathrm{d}\nu_\omega^-(x) - 1_{N^c}(\omega)\int f^-(\omega, x) df\nu_\omega^-(x),
\end{aligned}
$$

where the right-hand side is $\mathcal{F}$ measurable by Theorem A.1.16. This proves the second claim. For the third claim, we note that

$$
\begin{aligned}
&\int f(\omega, x)\,\mathrm{d}\lambda^+(\omega, x) \\
=& \int 1_{N^c}(\omega) f^+(\omega, x)\,\mathrm{d}\lambda^+(\omega, x) - \int 1_{N^c}(\omega) f^-(\omega, x)\,\mathrm{d}\lambda^+(\omega, x) \\
=& \int 1_{N^c}(\omega)\int f^+(\omega, x)\,\mathrm{d}\nu_\omega^+(x)\,\mathrm{d}P(\omega) - \int 1_{N^c}(\omega)\int f^-(\omega, x)\,\mathrm{d}\nu_\omega^+(x)\,\mathrm{d}P(\omega) \\
=& \int 1_{N^c}(\omega)\int f(\omega, x)\,\mathrm{d}\nu_\omega^+(x)\,\mathrm{d}P(\omega) = \int\int f(\omega, x)\,\mathrm{d}\nu_\omega^+(x)\,\mathrm{d}P(\omega),
\end{aligned}
$$

where all integrals are well-defined by Theorem A.1.16. By similar calculations, we obtain the same result for λ^-. Adding positive and negative parts, we obtain the proof of the third claim. □

Theorem A.1.18. *Let $\mathcal{H}$ be a nonempty family of random variables. There exists a variable X such that for all $Y \in \mathcal{H}$, $Y \leq X$ almost surely, and if X' is another variable with this property, then $X \leq X'$ almost surely. Furthermore, there exists a sequence $(X_n) \in \mathcal{H}$ such that $X = \sup_n X_n$. X is called the essential upper envelope of $\mathcal{H}$.*

Proof. See Theorem 1.13 of He et al. (1992). □

Lemma A.1.19. *Let X be some integrable variable. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. If $E1_F X \geq 0$ for all $F \in \mathcal{G}$, then $E(X|\mathcal{G}) \geq 0$ almost surely.*

Proof. Pick $n \in \mathbb{N}$ and define $F = (E(X|\mathcal{G}) \leq -\frac{1}{n})$. As $E(X|\mathcal{G})$ is $\mathcal{G}$ measurable, we have $F \in \mathcal{G}$ and therefore obtain $E1_F X = E1_F E(X|\mathcal{G}) \leq -\frac{1}{n}P(F)$. Therefore, $P(F) = 0$. By the continuity properties of probability measures, we conclude $P(E(X|\mathcal{G}) < 0) = 1$, so that $E(X|\mathcal{G}) \geq 0$ almost surely. □

Lemma A.1.20. *Let $X \geq 0$. It holds that X has mean zero if and only if X is almost surely zero.*

Proof. Clearly, X has mean zero if X is almost surely zero. Assume instead that X has mean zero. For any $n \in \mathbb{N}$, we have $EX \geq EX1_{(X \geq \frac{1}{n})} \geq \frac{1}{n}P(X \geq \frac{1}{n})$. Thus, we conclude $P(X \geq \frac{1}{n}) = 0$ for all n, and therefore $P(X > 0) = 0$, so that X is almost surely zero. □

Lemma A.1.21. *Let X and Y be two integrable variables. Assume that for all bounded variables ξ, it holds that $EX\xi = EY\xi$. Then X and Y are almost surely equal.*

Proof. Put $\xi = 1_{(X-Y>0)}$. Then $E(X-Y)1_{(X-Y>0)} = 0$, yielding that $(X-Y)1_{(X-Y>0)}$ is almost surely zero, so $(X - Y > 0)$ is a null set. Similarly, we obtain that $(X - Y < 0)$ is a null set, leading us to conclude that X and Y are almost surely equal. □

For the next results, recall that for any open set U in $\mathbb{R}^p$, $C^2(U)$ denotes the set of mappings $f : U \to \mathbb{R}$ such that all second-order partial derivatives of f exists and are continuous. Furthermore, $C^\infty(U)$ denotes the set of $f : U \to \mathbb{R}$ such that all partial derivatives of any order of f exists, and $C_c^\infty(U)$ denotes the set of elements in $C^\infty(U)$ which have compact support.

Theorem A.1.22. *Let $f \in C^2(\mathbb{R}^p)$, and let $x, y \in \mathbb{R}^p$. Assume that all second-order partial derivatives of f are uniformly continuous. It then holds that*

$$f(y) = f(x) + \sum_{i=1}^p \frac{\partial f}{\partial x_i}(x)(y_i - x_i) + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \frac{\partial^2 f}{\partial x_i \partial x_j}(x)(y_i - x_i)(y_j - x_j) + R_2(x, y),$$

where $R_2(x, y) = \sum_{i=1}^p \sum_{j=1}^p r_2^{ij}(y, x)(y_i - x_i)(y_j - x_j)$, and

$$r_2^{ij}(y, x) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\xi^{ij}(x, y)) - \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right),$$

where $\xi^{ij}(x, y)$ is some element on the line segment between x and y.

Proof. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(x + t(y - x))$. Note that $g(1) = f(y)$ and $g(0) = f(x)$. We will prove the theorem by applying the one-dimensional Taylor formula, see Apostol (1964) Theorem 7.6, to g. Clearly, $g \in C^2(\mathbb{R})$, and we obtain $g(1) = g(0) + g'(0) + \frac{1}{2}g''(s)$, where $0 \leq s \leq 1$. Applying the chain rule, we find $g'(t) = \sum_{i=1}^p \frac{\partial f}{\partial x_i}(x + t(y - x))(y_i - x_i)$ and $g''(t) = \sum_{i=1}^p \sum_{j=1}^p \frac{\partial^2 f}{\partial x_i \partial x_j}(x + t(y - x))(y_i - x_i)(y_j - x_j)$. Substituting and writing $\xi = x + s(y - x)$, we may conclude

$$f(y) = f(x) + \sum_{i=1}^p \frac{\partial f}{\partial x_i}(x)(y_i - x_i) + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi)(y_i - x_i)(y_j - x_j).$$

In particular, we find $R_2(x, y) = \sum_{i=1}^p \sum_{j=1}^p r_2^{ij}(y, x)(y_i - x_i)(y_j - x_j)$, where $r_2^{ij} : \mathbb{R}^2 \to \mathbb{R}$ is defined by putting

$$r_2^{ij}(y, x) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\xi) - \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right),$$

where ξ of course depends on x and y, as it is on the line segment between the two. This proves the result. □

Lemma A.1.23. *Let U be an open set in $\mathbb{R}^p$ and let $f \in C^2(U)$. Let $\varepsilon > 0$. With $\|\cdot\|$ denoting some norm on $\mathbb{R}^p$ and $d(x, y) = \|x - y\|$, put $F = \{x \in \mathbb{R}^p | d(x, U^c) \geq \varepsilon\}$. There exists $g \in C^2(\mathbb{R}^p)$ such that f and g agree on F.*

Proof. Let $G = \{x \in \mathbb{R}^p | d(x, U^c) \geq \frac{\varepsilon}{2}\}$ and $H = \{x \in \mathbb{R}^p | d(x, U^c) \geq \frac{\varepsilon}{4}\}$. We first prove that there exists a mapping $\chi \in C^\infty(\mathbb{R}^p)$ such that χ is one on F and zero on H^c. From Lemma 2.1 of Grubb (2008) and Section 0.B of Zimmer (1990), there exists a mapping $\psi \in C_c^\infty(\mathbb{R}^p)$ such that $\int_{\mathbb{R}^p} \psi(x) \,\mathrm{d}x = 1$ and ψ is zero outside of the open euclidean ball B centered at the origin with radius $\frac{\varepsilon}{4}$. Define $\chi : \mathbb{R}^p \to \mathbb{R}$ by putting $\chi(x) = \int_{\mathbb{R}^p} 1_G(y)\psi(x - y) \,\mathrm{d}y$, this is well-defined as ψ has compact support, and compact sets have finite Lebesgue measure. We claim that χ satisfies the requirements. Applying the methods of the proof of Proposition B.3 of Zimmer (1990), we find that $\chi \in C^\infty(\mathbb{R}^p)$. Note that by the translation invariance of Lebesgue measure, we have

$$\chi(x) = \int_{\mathbb{R}^p} 1_G(x-y)\psi(y)\,\mathrm{d}y = \int_B 1_G(x-y)\psi(y)\,\mathrm{d}y.$$

Now, given some $x \in F$, we find that for any $y \in B$, $d(x, U^c) \le d(x-y, U^c) + \|y\|$ and so $d(x-y, U^c) \ge d(x, U^c) - \|y\| \ge \varepsilon - \frac{\varepsilon}{4} \ge \frac{\varepsilon}{2}$. Thus, $x - y \in G$ and so $\chi(x) = \int_B \psi(y)\,\mathrm{d}y = 1$. Conversely, if $x \in H^c$, it holds that $d(x-y, U^c) \le d(x, U^c) + \|y\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$, so $x - y \notin G$, and $\chi(x) = 0$. Thus, χ is in $C^\infty(\mathbb{R}^p)$ and $\chi(x) = 1$ when $x \in F$ and $\chi(x) = 0$ when $x \in H^c$. We now define $g : \mathbb{R}^p \to \mathbb{R}$ by putting $g(x) = f(x)\chi(x)$ when $x \in U$ and $g(x) = 0$ otherwise. We claim that g satisfies the requirements of the lemma.

To see this, first note that when $x \in F$, $g(x) = f(x)\chi(x) = f(x)$, so g and f agree on F. Therefore, we merely need to check that g is C^2. To see this, note that on U, g is the product of an C^2 mapping and an C^∞ mapping, so g is C^2 on U. Conversely, as χ is zero on H^c, we find that g is in particular C^2 on H^c. As $H \subseteq U$, $U^c \subseteq H^c$ and so $\mathbb{R}^p = U \cup H^c$. Therefore, we conclude that g is in $C^2(\mathbb{R}^p)$, as desired. □

A.2 Càdlàg and finite variation mappings

In this section, we introduce càdlàg mappings and finite variation mappings and consider their connection to pairs of positive singular measures. These types of mappings will be important in our consideration of continuous-time stochastic processes of finite variation, of which the quadratic covariation will be a primary example. Consider a mapping $f : \mathbb{R}_+ \to \mathbb{R}$. We say that f is càdlàg if f is right-continuous on $\mathbb{R}_+$ and has limits from the left on $(0, \infty)$. For $t > 0$, we write $f(t-)$ for the limit of $f(s)$ with s converging upwards to t. By convention, $f(0-) = f(0)$. We define $\Delta f(t) = f(t) - f(t-)$.

Furthermore, for any mapping $f : \mathbb{R}_+ \to \mathbb{R}$, we define the variation of f on $[0, t]$ by $V_f(0) = 0$ and $V_f(t) = \sup |f(t_{k+1}) - f(t_k)|$, where the supremum is over all partitions of the type $0 = t_0 < \cdots < t_n = t$. We say that f is of finite variation on $[0, t]$ if $V_f(t)$ is finite. We say that f is of finite variation if $V_f(t)$ is finite for $t \ge 0$. We say that f is of bounded variation if $\sup_t V_f(t)$ is finite.

Finally, by **FV**, we denote the càdlàg mappings $f : \mathbb{R}_+ \to \mathbb{R}$ of finite variation, and by $\mathbf{FV}_0$

we define the elements of **FV** with initial value zero. By **cFV**, we denote the elements of **FV** which are continuous, and by $\mathbf{cFV}_0$, we denote the elements of $\mathbf{FV}_0$ which are continuous. Note that any monotone function has finite variation. Also, it holds that for any increasing function $f : \mathbb{R}_+ \to \mathbb{R}$ with initial value zero, $V_f(t) = f(t)$.

We begin by considering some results solely relating to càdlàg mappings.

Lemma A.2.1. *Let A be an infinite subset of $\mathbb{R}$. A contains either a strictly increasing sequence or a strictly decreasing subsequence.*

Proof. Assume that A contains no strictly decreasing sequence. Let (t_n) be a sequence of distinct elements in A, then (t_n) does not contain a strictly decreasing sequence either. Define $s_n = \inf_{k \geq n} t_k$, then (s_n) is increasing. Assume, expecting a contradiction, that (s_n) contains only finitely many different elements. Then (s_n) is constant from some point onwards, say N. For any $n \geq N$, we then have $\min\{t_n, \inf_{k\geq n+1} t_k\} = \inf_{k\geq n} t_k = \inf_{k \geq n+1} t_k$, so $t_n \geq t_k$ for $k \geq n+1$. We conclude that $(t_k)_{k\geq n}$ is decreasing. As the elements of (t_n) are distinct, $(t_k)_{k\geq n}$ is a strictly decreasing sequence of elements, a contradiction. We conclude that s_n contains infinitely many different numbers. In particular, there is a subsequence (s_{n_k}) which is strictly increasing.

Next, assume, again expecting a contradiction, that s_n does not attain its infimum. Then, for any $\varepsilon > 0$, there exists t_k with $k \geq n$ such that $s_n < t_k < s_n + \varepsilon$. In particular, there exists a decreasing sequence of distinct elements in $\{t_k\}_{k\geq n}$. Consisting of distinct elements, the sequence in fact constitutes a strictly decreasing sequence in A, a contradiction. We conclude that s_n attains its infimum. In particular, $s_{n_k} = t_{m_k}$ for some m_k. (t_{m_k}) is then a strictly increasing sequence in A, since (s_{n_k}) is strictly increasing. □

Lemma A.2.2. *If $f : \mathbb{R}_+ \to \mathbb{R}$ is càdlàg, then f is bounded on compact sets.*

Proof. It suffices to prove that f is bounded on $[0, t]$ for $t > 0$. Assume contrarily that there is some $t > 0$ such f is not bounded on $[0, t]$. There exists a sequence (s_n) such that $f(s_n)$ is unbounded. In particular, (s_n) is infinite and we may assume that $|f(s_n)|$ has no convergent subsequence. By Lemma A.2.1, (s_n) has either a strictly increasing subsequence (s_{n_k}) or a strictly decreasing subsequence (s_{n_k}). In both cases, $f(s_{n_k})$ is convergent by the càdlàg property, which is in contradiction with the assumption that $|f(s_n)|$ has no convergent subsequence. We conclude that f is bounded on $[0, t]$. □

Lemma A.2.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a càdlàg mapping and let $t \geq 0$. For any $\varepsilon > 0$, $|\Delta f(s)| \geq \varepsilon$ only for finitely many $s \in [0, t]$. In particular, f only has countably many jumps.*

Proof. Let $\varepsilon > 0$ and $t > 0$, and assume contrarily that there is a sequence (t_n) in $[0,t]$ such that $|\Delta f(t_n)| \geq \varepsilon$ for all n and such that all the numbers t_n are distinct. By Lemma A.2.1, (t_n) has a strictly increasing convergent subsequence or (t_n) has a strictly decreasing convergent subsequence.

Assume first that (t_n) has a strictly increasing subsequence (t_{n_k}) with limit s. For any $\delta > 0$, there exists k such that $s - \delta < t_{n_k} < s$. As $|\Delta f(t_{n_k})| \geq \varepsilon$, we must have either $|f(t_{n_k}) - f(s-)| \geq \frac{\varepsilon}{2}$ or $|f(t_{n_k}-) - f(s-)| \geq \frac{\varepsilon}{2}$. In any case, we see that there is u with $s - \delta < u < s$ such that $|f(u) - f(-s)| \geq \frac{\varepsilon}{4}$. As δ was arbitrary, this is a contradiction. We conclude that (t_n) cannot have a strictly increasing subsequence. Analogously, we may prove that (t_n) cannot have a strictly decreasing subsequence. We have obtain a contradiction and conclude that for any $\varepsilon > 0$, $|\Delta f(s)| \geq \varepsilon$ only for finitely many $s \in [0,t]$. As we can write $\{t \geq 0 | \Delta f(t) \neq 0\} = \cup_{n=1}^{\infty} \cup_{k=1}^{\infty} \{t \in [0,n] | |\Delta f(t)| \geq \frac{1}{k}\}$, we find in particular that f only has countably many jumps. □

Lemma A.2.4. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be càdlàg. Fix $\varepsilon > 0$. For any $t > 0$, there exists $0 = t_0 < \cdots < t_p = t$ such that for $i \leq p$, $\sup_{t_{i-1} \leq s,r < t_i} |f(s) - f(r)| \leq \varepsilon$.*

Proof. Let $\varepsilon > 0$ be given and let A be the set of $t > 0$ such that the finitely many numbers exists. We wish to prove that $A = (0, \infty)$. To this end, first consider some $t \in A$ and let $0 < u \leq t$, we will prove that $u \in A$. Let $0 = t_0 < \cdots < t_p = t$ be such that $\sup_{t_{i-1} \leq s,r < t_i} |f(s) - f(r)| \leq \varepsilon$ for $i \leq p$. Let $u_i = t_i \wedge u$ for $i \leq p$. Clearly, for i such that $t_i < u$, $\sup_{u_{i-1} \leq s,r < u_i} |f(s) - f(r)| \leq \varepsilon$. Let q be the first i such that $u_i = u$, we then have $\sup_{u_{i-1} \leq s,r < u_i} |f(s) - f(r)| \leq \sup_{t_{i-1} \leq s,r < t_i} |f(s) - f(r)| \leq \varepsilon$. Therefore, $\{u_0, \ldots, u_q\}$ is a partition of $[0,u]$ such that $\sup_{u_{i-1} \leq s,r < u_i} |f(s) - f(r)| \leq \varepsilon$. Thus, $u \in A$. We conclude that whenever $t \in A$, $u \in A$ as well for all u with $0 < u \leq t$. Next, also note that as f is right-continuous at zero, it is immediate that A contains $(0, \delta]$ for some $\delta > 0$. Combining our observations, we find that in order to show that $A = \mathbb{R}_+$, it suffices to show that $\sup A$ is infinite.

To this end, assume that $\sup A$ is finite, we wish to obtain a contradiction. Let τ denote the supremum. By our previous results, we then have $[0, \tau) \subseteq A$. Let $x_{\tau-}$ denote the left limit of X at τ. Since X has a left-limit and is right-continuous at τ, there exists $\delta > 0$ such that whenever $t \in [\tau - \delta, \tau)$, we have $|f(\tau-) - f(t)| \leq \frac{\varepsilon}{2}$, and when $t \in [\tau, \tau + \delta]$, $|f(\tau) - f(t)| \leq \frac{\varepsilon}{2}$. In particular, for any $s, r \in [\tau - \delta, \tau)$ we have $|f(s) - f(r)| \leq |f(s) - f(\tau-)| + |f(\tau-) - f(r)| \leq \varepsilon$, and for any $s, r \in [\tau, \tau + \delta)$, we have $|f(s) - f(r)| \leq |f(s) - f(\tau)| + |f(\tau) - f(r)| \leq \varepsilon$.

Now, as $\tau - \delta \in A$, we may pick finitely many numbers $0 = t_0 < \cdots < t_p = \tau - \delta$ such that $\sup_{t_{i-1}\le s,r<t_i}|f(s)-f(r)|\le\varepsilon$ for $i\le p$. Putting $u_i=t_i$ for $i\le p$, $u_{p+1}=\tau$ and $u_{p+2}=\tau+\delta$, we then obtain that that the sequence $u_0,\ldots,u_{p+2}$ shows that $\tau+\delta\in A$. This is in contradiction with the fact that τ is the finite supremum of A, and we conclude that the supremum must be infinite. This concludes the proof. □

Lemma A.2.5. *Let $f:\mathbb{R}_+\to\mathbb{R}$ be càdlàg. Let $t>0$ and let $(t_k^n)_{k\le K_n}$ be a sequence of partitions of $[0,t]$ with mesh tending to zero. Let A be some subset of $[0,t]$. Defining $I_n=\{1\le k\le K_n \mid A\cap(t_{k-1}^n,t_k^n]=\emptyset\}$, it holds that*

$$\limsup_{n\to\infty}\max_{k\in I_n}\sup_{t_{k-1}^n\le r,s\le t_k^n}|f(s)-f(r)|\le 3\sup_{x\in[0,t]\setminus A}|\Delta f(x)|.$$

Proof. Fix $t>0$ and consider $\eta>0$. Fix n and $k\in I_n$. Using that f is right-continuous, pick $\delta>0$ with $t_{k-1}^n+\delta<t_k^n$ such that $|f(s)-f(t_{k-1}^n)|\le\eta$ whenever $s\in[t_{k-1}^n,t_{k-1}^n+\delta]$. Then, for $s\in[t_{k-1}^n,t_{k-1}^n+\delta]$, we obtain $|f(s)-f(t_{k-1}^n)|\le\eta$, and for $s\in(t_{k-1}^n+\delta,t_k^n]$, we obtain $|f(s)-f(t_{k-1}^n)|\le\eta+|f(s)-f(t_{k-1}^n+\delta)|$. This shows that

$$\sup_{t_{k-1}^n\le r,s\le t_k^n}|f(s)-f(r)|\le\eta+\sup_{t_{k-1}^n<r,s\le t_k^n}|f(s)-f(r)|,$$

and as $\eta>0$ was arbitrary, $\sup_{t_{k-1}^n\le r,s\le t_k^n}|f(s)-f(r)|\le\sup_{t_{k-1}^n<r,s\le t_k^n}|f(s)-f(r)|$. Next, note that

$$\sup_{t_{k-1}^n<r,s\le t_k^n}|f(s)-f(r)|\le\sup_{t_{k-1}^n<r,s<t_k^n}|f(s)-f(r)|+\sup_{t_{k-1}^n<r<t_k^n}|f(t_k^n)-f(r)|.$$

As $A\cap(t_{k-1}^n,t_k^n]=\emptyset$, we in particular have $t_k^n\notin A$, so

$$\begin{aligned}\sup_{t_{k-1}^n<r<t_k^n}|f(t_k^n)-f(r)| &\le |\Delta f(t_k^n)|+\sup_{t_{k-1}^n<r<t_k^n}|f(t_k^n-)-f(r)|\\ &\le \sup_{x\in[0,t]\setminus A}|\Delta f(x)|+\sup_{t_{k-1}^n<r,s<t_k^n}|f(s)-f(r)|,\end{aligned}$$

so that all in all, we obtain

$$\max_{k\in I_n}\sup_{t_{k-1}^n\le r,s\le t_k^n}|f(s)-f(t_{k-1}^n)|\le\sup_{x\in[0,t]\setminus A}|\Delta f(x)|+2\max_{k\in I_n}\sup_{t_{k-1}^n<r,s<t_k^n}|f(s)-f(r)|.$$

We consider the limes superior of the latter term. Again, fix $\eta>0$. By Lemma A.2.4, there is a partition $0=s_0<\cdots<s_p=t$ with the property that $\max_{s_{i-1}\le r,s<s_i}|f(s)-f(r)|\le\eta$ for all $i\le p$. Now let n be so large that for each $k\le K_n$, each interval $[t_{k-1}^n,t_k^n]$ contains at most one element of $\{s_0,\ldots,s_p\}$, this is possible as the mesh of the partitions tend to zero. Let $k\in I_n$. If (t_{k-1}^n,t_k^n) does not contain any of the points in $\{s_0,\ldots,s_p\}$, then (t_{k-1}^n,t_k^n) is included in $[s_{i-1},s_i)$ for some i and so $\sup_{t_{k-1}^n<r,s<t_k^n}|f(s)-f(r)|\le\eta$. Contrarily, assume

that (t^n_{k-1}, t^n_k) contains some s_i. Let $t^n_{k-1} < r, s < t^n_k$. If $t^n_{k-1} < r, s < s_i$ or $s_i \leq r, s < t^n_k$, we obtain $|f(s) - f(r)| \leq \eta$. If instead $t^n_{k-1} < r < s_i \leq s < t^n_k$, we obtain

$$\begin{aligned} |f(s) - f(r)| &\leq |f(s) - f(s_i)| + |f(s_i) - f(r)| \\ &\leq |f(s) - f(s_i)| + |\Delta f(s_i)| + |f(s_i-) - f(r)| \\ &\leq 2\eta + |\Delta f(s_i)|, \end{aligned}$$

and similarly for $t^n_{k-1} < s < s_i \leq r < t^n_k$. As we have assumed $s_i \in (t^n_{k-1}, t^n_k)$ and $k \in I_n$, we have $s_i \notin A$, so $|\Delta f(s_i)| \leq \sup_{x \in [0,t] \setminus A} |\Delta f(x)|$. All in all, we obtain that for n large enough and for all $k \in I_n$ for such n, $\sup_{t^n_{k-1} < r,s < t^n_k} |f(s) - f(r)| \leq 2\eta + \sup_{x \in [0,t] \setminus A} |\Delta f(x)|$. From this, we conclude

$$\limsup_{n \to \infty} \max_{k \in I_n} \sup_{t^n_{k-1} < r,s < t^n_k} |f(s) - f(r)| \leq \sup_{x \in [0,t] \setminus A} |\Delta f(x)|,$$

and combining this with our earlier results, we obtain the desired result. □

Lemma A.2.6. *Let (f_n) be a sequence of bounded càdlàg mappings from $\mathbb{R}_+$ to $\mathbb{R}$. If (f_n) is Cauchy in the uniform norm, there is a bounded càdlàg mapping f from $\mathbb{R}_+$ to $\mathbb{R}$ such that $\sup_{t \geq 0} |f_n(t) - f(t)|$ tends to zero. In this case, it holds that $\sup_{t \geq 0} |f_n(t-) - f(t-)|$ and $\sup_{t \geq 0} |\Delta f_n(t) - \Delta f(t)|$ tends to zero as well.*

Proof. Assume that for any $\varepsilon > 0$, for n and m large enough, $\sup_{t \geq 0} |f_n(t) - f_m(t)| \leq \varepsilon$. This implies that $(f_n(t))_{n \geq 1}$ is Cauchy for any $t \geq 0$, therefore convergent. Let $f(t)$ be the limit. Now note that as (f_n) is Cauchy in the uniform norm, (f_n) is bounded in the uniform norm, and therefore $\sup_{t \geq 0} |f(t)| \leq \sup_{n \geq 1} \sup_{t \geq 0} |f_n(t)|$, so f is bounded as well. In order to obtain uniform convergence, let $\varepsilon > 0$. Let k be such that for $m, n \geq k$, $\sup_{t \geq 0} |f_n(t) - f_m(t)| \leq \varepsilon$. Fix $t \geq 0$, we then obtain for $n \geq k$ that $|f(t) - f_n(t)| = \lim_m |f_m(t) - f_n(t)| \leq \varepsilon$. Therfore, $\sup_{t \geq 0} |f(t) - f_n(t)| \leq \varepsilon$, and so f_n converges uniformly to f.

We now show that f is càdlàg. Let $t \geq 0$, we will show that f is right-continuous at t. Take $\varepsilon > 0$ and take n so that $\sup_{t \geq 0} |f(t) - f_n(t)| \leq \varepsilon$. Let $\delta > 0$ be such that $|f_n(t) - f_n(s)| \leq \varepsilon$ for $s \in [t, t + \delta]$, then $|f(t) - f(s)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f(s)| \leq 3\varepsilon$ for such s. Therefore, f is right-continuous at t. Now let $t > 0$, we claim that f has a left limit at t. First note that for n and m large enough, it holds for any $t > 0$ that $|f_n(t-) - f_m(t-)| \leq \sup_{t \geq 0} |f_n(t) - f_m(t)|$. Therefore, the sequence $(f_n(t-))_{n \geq 1}$ is Cauchy, and so convergent to some limit $\xi(t)$. Now let $\varepsilon > 0$ and take n so that $\sup_{t \geq 0} |f(t) - f_n(t)| \leq \varepsilon$ and $|f_n(t-) - \xi(t)| \leq \varepsilon$. Let $\delta > 0$ be such that $t - \delta \geq 0$ and such that whenever $s \in [t - \delta, t)$, $|f_n(s) - f_n(t-)| \leq \varepsilon$. Then $|f(s) - \xi(t)| \leq |f(s) - f_n(s)| + |f_n(s) - f_n(t-)| + |f_n(t-) - \xi(t)| \leq 3\varepsilon$ for any such s. Therefore, f has a left limit at t. This shows that f is càdlàg.

Finally, we have for any $t > 0$ and any sequence (s_n) converging strictly upwards to t that $|f(t-) - f_n(t-)| = \lim_m |f(s_m) - f_n(s_n)| \le \sup_{t\ge 0} |f(t) - f_n(t)|$, so $\sup_{t\ge 0} |f(t-) - f_n(t-)|$ converges to zero as well. Therefore, $\sup_{t\ge 0} |\Delta f(t) - \Delta f_n(t)|$ converges to zero as well. □

Lemma A.2.7. *Let (f_n) be a sequence of nonnegative increasing càdlàg mappings from $\mathbb{R}_+$ to $\mathbb{R}$. Assume that $\sum_{n=1}^\infty f_n$ converges pointwise to some mapping f from $\mathbb{R}_+ \to \mathbb{R}$. Then, the convergence is uniform on compacts, and f is a nonnegative increasing càdlàg mapping. If $f(t)$ has a limit as t tends to infinity, the convergence is uniform on $\mathbb{R}_+$.*

Proof. Fix $t \ge 0$. For $m \ge n$, we have

$$\sup_{0\le s\le t} \left| \sum_{k=1}^m f_k(s) - \sum_{k=1}^n f_k(s) \right| = \sup_{0\le s\le t} \sum_{k=n+1}^m f_k(s) = \sum_{k=n+1}^m f_k(t),$$

which tends to zero as m and n tend to infinity. Therefore, $(\sum_{k=1}^n f_k)$ is uniformly Cauchy on $[0,t]$, and so has a càdlàg limit on $[0,t]$. As this limit must agree with the pointwise limit, we conclude that $\sum_{k=1}^n f_k$ converges uniformly on compacts to f, and therefore f is nonnegative, increasing and càdlàg.

It remains to consider the case where $f(t)$ has a limit $f(\infty)$ as t tends to infinity. In this case, we find that $\lim_t f_n(t) \le \lim_t f(t) = f(\infty)$, so $f_n(t)$ has a limit $f_n(\infty)$ as t tends to infinity as well. Fixing $n \ge 1$, we have

$$\sum_{k=1}^n f_k(\infty) \quad = \quad \sum_{k=1}^n \lim_{t\to\infty} f_k(t) = \lim_{t\to\infty} \sum_{k=1}^n f_k(t) \le \lim_{t\to\infty} f(t) = f(\infty).$$

Therefore, $(f_k(\infty))$ is absolutely summable. As we have

$$\sup_{0\le t} \left| \sum_{k=1}^m f_k(t) - \sum_{k=1}^n f_k(t) \right| = \sup_{0\le t} \sum_{k=n+1}^m f_k(t) = \sum_{k=n+1}^m f_k(\infty),$$

we find that $(\sum_{k=1}^n f_k)$ is uniformly Cauchy on $\mathbb{R}_+$, and therefore uniformly convergent. As the limit must agree with the pointwise limit, we conclude that f_n converges uniformly to f on $\mathbb{R}_+$. This concludes the proof. □

Next, we consider càdlàg finite variation mappings, in particular introducing the integral with respect to such a mapping.

Lemma A.2.8. *If $f \in \mathbf{FV}_0$, then V_f is càdlàg and $\Delta V_f(t) = |\Delta f(t)|$. If $f \in \mathbf{cFV}_0$, V_f is continuous.*

Proof. That V_f is càdlàg when $f \in \mathbf{FV}_0$ and V_f is continuous when $f \in \mathbf{cFV}_0$ follows from Theorem 13.9 of Carothers (2000). It remains to prove that $\Delta V_f = |\Delta f(t)|$. We may restrict our attention to $t > 0$. We will show $V_f(t) = V_f(t-) + |\Delta f(t)|$. First, note that $V_f(t-) = \sup \sum_{k=1}^n |f(t_k) - f(t_{k-1})|$, where (t_k) is a partition of $[0, s]$ for some $s < t$. Now consider a sequence of partitions of $[0, t]$, $0 = t_0^m < \ldots < t_{n_m}^m = t$, such that we have $V_f(t) = \lim_m \sum_{k=1}^{n_m} |f(t_k^m) - f(t_{k-1}^m)|$. We can assume without loss of generality that $t_{n_m-1}^m$ tends to t. We then have

$$\begin{aligned} V_f(t) &= \lim_m |f(t) - f(t_{n_m-1}^m)| + \sum_{k=1}^{n_m-1} |f(t_k^m) - f(t_{k-1}^m)| \\ &\leq V_f(t-) + \lim_m |f(t) - f(t_{n_m-1}^m)| = V_f(t-) + |\Delta f(t)|. \end{aligned}$$

On the other hand, let (P_n) be a sequence of partitions of $[0, t)$, $P_m = (t_0^m, \ldots, t_{n_m}^m)$, such that $V_f(t-) = \lim_m \sum_{k=1}^{n_m} |f(t_k^m) - f(t_{k-1}^m)|$. Here, we can assume without loss of generality that $t_{n_m}^m$ tends to t. We then obtain

$$V_f(t-) + |\Delta f(t)| = \lim_m |f(t) - f(t_{n_m}^m)| + \lim_m \sum_{k=1}^{n_m} |f(t_k^m) - f(t_{k-1}^m)| \leq V_f(t).$$

This proves the lemma. □

Theorem A.2.9. *Let $f \in \mathbf{FV}_0$. There is a unique decomposition $f = f^+ - f^-$ such that f^+ and f^- are increasing functions in $\mathbf{FV}_0$ with the property that there exists two unique positive singular measures μ_f^+ and μ_f^- with zero point mass at zero such that for any $0 \leq a \leq b$, $\mu_f^+(a, b] = f^+(b) - f^+(a)$ and $\mu_f^-(a, b] = f^-(b) - f^-(a)$. The decomposition is given by $f^+ = \frac{1}{2}(V_f + f)$ and $f^- = \frac{1}{2}(V_f - f)$. In particular, the measures μ_f^+ and μ_f^- are finite on bounded intervals, and $(\mu_f^+ + \mu_f^-)(a, b] = V_f(b) - V_f(a)$.*

Proof. We first show that the explicit construction of f^+ and f^- satisfies the properties required. It is immediate that f^+ and f^- are increasing and zero at zero, and so, as monotone function are of finite variation, we conclude that f^+ and f^- are in $\mathbf{FV}_0$, the càdlàg property being a consequence of Lemma A.2.8. By Theorem 1.4.4 of Ash (2000), there exists unique nonnegative measures μ_f^+ and μ_f^- with zero point mass at zero such that for any $0 \leq a \leq b$, $\mu_f^+(a, b] = f^+(b) - f^+(a)$ and $\mu_f^-(a, b] = f^-(b) - f^-(a)$. Then $(\mu_f^+ + \mu_f^-)(a, b] = V_f(b) - V_f(a)$ as well. It remains to prove that μ_f^+ and μ_f^- are singular, and to this end, it suffices to prove that the measures are singular on $[0, t]$ for any $t \geq 0$.

To do so, fix $t \geq 0$. Put $\mu_f^t = \mu_f^+ - \mu_f^-$ on $\mathcal{B}_t$, then μ_f^t is a signed measure on $\mathcal{B}_t$, and for any $0 \leq a \leq b \leq t$, $\mu_f^t(a, b] = f(b) - f(a)$. We consider the total variation of μ_f^t. Fix

$0 \leq a \leq b \leq t$ and let $\mathcal{A}$ be the set of finite unions of intervals of the form $(c,d]$ with $a \leq c \leq d \leq b$, $\mathcal{A}$ is an algebra generating the Borel-σ-algebra on $(a,b]$. Lemma A.1.6 shows that we have $|\mu_f^t|(a,b] = \sup \sum_{n=1}^k |\mu_f^t(A_n)|$, where (A_n) is a finite disjoint partition of $(a,b]$ with elements from $\mathcal{A}$. In particular, we obtain $|\mu_f^t|(a,b] \leq V_f(t)$, and as we trivially have $V_f(t) \leq |\mu_f^t|(a,b]$, we have equality. Thus, $|\mu_f^t|(a,b] = V_f(b) - V_f(a)$. Let $(\mu_f^t)^+$ and $(\mu_f^t)^-$ be the Jordan-Hahn decomposition of Theorem A.1.5, we then obtain

$$(\mu_f^t)^+(a,b] = \tfrac{1}{2}(|\mu_f^t|(a,b] + \mu_f^t(a,b]) = \tfrac{1}{2}(V_f(b) - V_f(a) + f(b) - f(a)) = \mu_f^+(a,b],$$

and so we find that $(\mu_f^t)^+$ and μ_f^+ agree on $\mathcal{B}_t$. Analogously, $(\mu_f^t)^-$ and μ_f^- agree on $\mathcal{B}_t$ as well. As the components of the Jordan-Hahn decomposition are singular, we conclude that μ_f^+ and μ_f^- are singular on $[0,t]$, and so μ_f^+ and μ_f^- are singular measures.

It remains to prove uniqueness. Assume that $f = g^+ - g^-$ is another decomposition with the same properties. Let ν_f^+ and ν_f^- be the two corresponding singular positive measures. As earlier, we may then define $\nu_f^t = \nu_f^+ - \nu_f^-$ on $\mathcal{B}_t$. Then ν_f^t and μ_f^t are equal, and so in particular, we have the Jordan-Hahn decompositions $\mu_f^t = \mu_f^+ - \mu_f^-$ and $\mu_f^t = \nu_f^t = \nu_f^+ - \nu_f^-$ on $\mathcal{B}_t$. By uniqueness of the decomposition, we conclude $\mu_f^+ = \nu_f^+$ and $\mu_f^- = \nu_f^-$, and so $f^+ = g^+$ and $f^- = g^-$, proving uniqueness. □

Theorem A.2.9 shows that càdlàg finite variation mappings with initial value zero correspond to pairs of positive singular measures. As stated in the theorem, for any $f \in \mathbf{FV}_0$, we denote by f^+ and f^- the increasing and decreasing parts of f, given by $f^+ = \frac{1}{2}(V_f + f)$ and $f^- = \frac{1}{2}(V_f - f)$. Furthermore, we denote by μ_f^+ and μ_f^- the two corrresponding positive singular measures, and we put $|\mu_f| = \mu^+ + \mu^-$ and call $|\mu_f|$ the total variation measure of f. By Theorem A.2.9, $|\mu_f|$ is the measure induced by the increasing function V_f using Theorem 1.4.4 of Ash (2000). As μ_f^+ and μ_f^- has finite mass on bounded intervals, so does $|\mu_f|$, in particular we have $|\mu_f|([0,t]) = V_f(t)$ according to Theorem A.2.9. Also note that if f is increasing, $V_f = f$ and so μ^- is zero.

These results lead to a concept of integration with respect to càdlàg functions of finite variation. Let $f \in \mathbf{FV}_0$ and let $h : \mathbb{R}_+ \to \mathbb{R}$ be some measurable function. We say that h is integrable with respect to f if $\int_0^t |h(s)| \,\mathrm{d}|\mu_f|_s$ is finite for all $t \geq 0$, and in the affirmative, we put $\int_0^t h(s)\,\mathrm{d}f(s) = \int_0^t h(s)\,\mathrm{d}(\mu_f^+)_s - \int_0^t h(s)\,\mathrm{d}(\mu_f^-)_s$ and call $\int_0^t h(s)\,\mathrm{d}f(s)$ the integral of h with respect to f over $[0,t]$. Furthermore, we denote by $\int_0^t h(s)|\,\mathrm{d}f_s|$ the integral $\int_0^t h(s)\,\mathrm{d}|\mu_f|_s$. Next, we consider some further properties of finite variation mappings and their integrals.

Lemma A.2.10. *Let $f \in \mathbf{FV}_0$. Then $|f(t)| \leq V_f(t)$ for all $t \geq 0$.*

Proof. As $\{0,t\}$ is a partition of $[0,t]$, $|f(t)| = |f(t)-f(0)| \leq \sup |f(t_k)-f(t_{k-1})| = V_f(t)$. $\square$

Lemma A.2.11. *Let $f \in \mathbf{FV}_0$. If $g : \mathbb{R}^+ \to \mathbb{R}$ is continuous, then g is integrable with respect to f, and for each $t \geq 0$ and sequence of partitions $0 = t_0 < \cdots < t_n = t$, the Riemann sums $\sum_{k=1}^n g(t_{k-1})(f(t_k) - f(t_{k-1}))$ converge to $\int_0^t g(s)\,\mathrm{d}f(s)$ as the mesh $\max_{k\leq n} |t_k - t_{k-1}|$ of the partition tends to zero.*

Proof. As g is continuous, g is bounded on compacts, and therefore g is integrable with respect to f. We have

$$\begin{aligned}\sum_{k=1}^n g(t_{k-1})(f(t_k) - f(t_{k-1})) &= \sum_{k=1}^n g(t_{k-1})\mu_f^+((t_{k-1},t_k]) - \sum_{k=1}^n g(t_{k-1})\mu_f^-((t_{k-1},t_k]) \\ &= \int_0^t s_n \,\mathrm{d}\mu_f^+ - \int_0^t s_n \,\mathrm{d}\mu_f^-,\end{aligned}$$

where $s_n = \sum_{k=1}^n g(t_{k-1}) 1_{(t_{k-1},t_k]}$. As the mesh tends to zero, s_n tends to g by the continuity of g. As g is bounded on $[0,t]$, two applications of the dominated convergence theorem yields the result. $\square$

Lemma A.2.12. *Let $f \in \mathbf{FV}_0$ and let h be integrable with respect to f. It then holds that $|\int_0^t h(s)\,\mathrm{d}f_s| \leq \int_0^t |h(s)||\,\mathrm{d}f_s|$.*

Proof. We find

$$\begin{aligned}\left|\int_0^t h(s)\,\mathrm{d}f_s\right| &= \left|\int_0^t h(s)\,\mathrm{d}\mu_f^+ - \int_0^t h(s)\,\mathrm{d}\mu_f^-\right| \leq \left|\int_0^t h(s)\,\mathrm{d}\mu_f^+\right| + \left|\int_0^t h(s)\,\mathrm{d}\mu_f^-\right| \\ &\leq \int_0^t |h(s)|\,\mathrm{d}\mu_f^+ + \int_0^t |h(s)|\,\mathrm{d}\mu_f^- = \int_0^t |h(s)|\,\mathrm{d}|\mu_f|,\end{aligned}$$

and the latter is what we denote by $\int_0^t |h(s)||\,\mathrm{d}f_s|$. $\square$

Lemma A.2.13 (Integration by parts)**.** *Let $f, g \in \mathbf{FV}$, then for any $t \geq 0$,*

$$f(t)g(t) = f(0)g(0) + \int_0^t f(s-)\,\mathrm{d}g_s + \int_0^t g(s-)\,\mathrm{d}f_s + \sum_{0<s\leq t} \Delta f(s)\Delta g(s),$$

where the sum converges absolutely.

Proof. To obtain the absolute convergence of the sum, we note that Lemma A.2.8 and the fact that V_f and V_g are both increasing functions allows us to conclude

$$\sum_{0<s\leq t} |\Delta f(s)\Delta g(s)| \leq \sum_{0<s\leq t} \Delta V_f(s)\Delta V_g(s) \leq V_f(t) \sum_{0<s\leq t} \Delta V_g(s) \leq V_f(t)V_g(t).$$

For the proof of the integration-by-parts formula, see Section IV.18 of Rogers & Williams (2000b). □

Lemma A.2.14. *Let $f \in \mathbf{FV}_0$ be increasing, let μ_f be the positive measure induced by f and let $g : \mathbb{R}_+ \to \mathbb{R}$ be measurable. Define $\beta : \mathbb{R}_+ \to \mathbb{R}$ by $\beta(s) = \inf\{t \geq 0 | f(t) \geq s\}$. Then g is integrable with respect to μ_f if and only if $s \mapsto g(\beta(s))1_{(\beta(s)<\infty)}$ is Lebesgue integrable, and in the affirmative case,*

$$\int_0^\infty g(t)\,\mathrm{d}\mu_f(t) = \int_0^\infty g(\beta(s))1_{(\beta(s)<\infty)}\,\mathrm{d}s.$$

Proof. First note that the conclusion is well-defined, since β is increasing and thus measurable. As the set of g such that the result holds is a vector space stable under pointwise increasing convergence of nonnegative mappings, it will suffice to prove the results for the mapping $g = 1_{[0,u]}$, $u \geq 0$. Therefore, let $u \geq 0$. First off, we have $\int_0^\infty 1_{[0,u]}(t)\,\mathrm{d}\mu_f(t) = f(u)$. To analyze the expression containing β, first note that since $1_{[0,u]}$ is well-defined in infinity as zero, we can write $\int_0^\infty 1_{[0,u]}(\beta(s))1_{(\beta(s)<\infty)}\,\mathrm{d}s = \int_0^\infty 1_{[0,u]}(\beta(s))\,\mathrm{d}s$. Next, note that if $\beta(s) < t$, then $f(t) \geq s$. And if $\beta(s) = t$, there is a sequence t_n converging downwards to t such that $f(t_n) \geq s$. By right-continuity, $f(t) \geq s$. In total, we conclude that if $\beta(s) \leq t$, then $f(t) \geq s$. Likewise, if $f(t) \geq s$, then $\beta(s) \leq t$. We therefore obtain

$$\int_0^\infty 1_{[0,u]}(\beta(s))\,\mathrm{d}s = \int_0^\infty 1_{[0,f(u)]}(s)\,\mathrm{d}s = f(u),$$

as desired. □

Lemma A.2.15. *If $f \in \mathbf{FV}_0$, then V_f can be written as $V_f(t) = \sup \sum_{k=1}^n |F(t_k) - F(t_{k-1})|$, where the supremum is taken over partitions in $\mathbb{Q}_+ \cup \{t\}$.*

Proof. It will suffice to prove that for any $\varepsilon > 0$ and partition $(t_0, \ldots, t_n)$, there exists another partition $(q_0, \ldots, q_n)$ such that $|\sum_{k=1}^n |f(t_k) - f(t_{k-1})| - \sum_{k=1}^n |f(q_k) - f(q_{k-1})|| < \varepsilon$. To this end, choose δ parrying $\frac{\varepsilon}{2n}$ for the right-continuity of F in the points $t_0, \ldots, t_{n-1}$. Let, for $k < n$, q_k be some rational with $q_k \geq t_k$ and $|q_k - t_k| \leq \delta$, and put $q_n = t_n$. It then holds that $|(f(t_k) - f(t_{k-1})) - (f(q_k) - f(q_{k-1}))| \leq \frac{\varepsilon}{n}$, and since $|\cdot|$ is a contraction, this implies $||f(t_k) - f(t_{k-1})| - |f(q_k) - f(q_{k-1})|| \leq \frac{\varepsilon}{n}$, finally yielding

$$\begin{aligned} &\left| \sum_{k=1}^n |f(t_k) - f(t_{k-1})| - \sum_{k=1}^n |f(q_k) - f(q_{k-1})| \right| \\ \leq\ & \sum_{k=1}^n ||f(t_k) - f(t_{k-1})| - |f(q_k) - f(q_{k-1})|| \leq \varepsilon, \end{aligned}$$

proving the result. □

Lemma A.2.16. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be càdlàg. If f has finite variation, $\sum_{0<s\leq t} |\Delta f(s)|^p$ is finite for all $p \geq 1$ and $t \geq 0$. If f has bounded variation, $\sum_{0<t} |\Delta f(t)|^p$ is finite as well.*

Proof. We first prove the claim in the case where $p = 1$. First assume that f has finite variation and fix $t \geq 0$. By Lemma A.2.3, f has only countably many jumps on $[0, t]$, let (t_n) be an enumeration of the jumps of f on $[0, t]$. Fix n, let $\varepsilon > 0$ and let $s_1, \ldots, s_n$ be the ordered values of $t_1, \ldots, t_n$. Defining $s_0 = 0$, for each $k \leq n$, take $u_k \in (s_{k-1}, s_k)$ such that $||f(s_k) - f(u_k)| - |\Delta f(s_k)|| \leq \frac{\varepsilon}{n}$. We then have

$$\sum_{k=1}^{n} |\Delta f(t_k)| = \sum_{k=1}^{n} |\Delta f(s_k)| \leq \varepsilon + \sum_{k=1}^{n} |f(s_k) - f(u_k)| \leq V_f(t) + \varepsilon,$$

as the set $(s_0, u_1, s_1, \ldots, u_n, s_n, t)$ constitutes a partition of $[0, t]$. As $\varepsilon > 0$ was arbitrary, we conclude $\sum_{k=1}^{n} |\Delta f(t_k)| \leq V_f(t)$. As n was arbitrary, this shows $\sum_{0<s\leq t} |\Delta f(s)| \leq V_f(t)$, so the sum converges. In the case where f has bounded variation, we can apply the same argument to obtain $\sum_{0<t} |\Delta f(t)| \leq V_f(\infty)$, so the sum also converges in this case.

Now let $p > 1$. In the case where f has finite variation, fix $t \geq 0$ and note that by Lemma A.2.3, f only have finitely many jumps of magnitude larger than 1, say $t_1, \ldots, t_n$. We then obtain

$$\begin{aligned} \sum_{0<s\leq t} |\Delta f(s)|^p &= \sum_{k=1}^{n} |\Delta f(t_k)|^p + \sum_{0<s\leq t} |\Delta f(s)|^p 1_{(|\Delta f(s)|\leq 1)} \\ &\leq \sum_{k=1}^{n} |\Delta f(t_k)|^p + \sum_{0<s\leq t} |\Delta f(s)|, \end{aligned}$$

which is finite by what we already proved. Finally, assume that f has bounded variation. In this case, f can only have finitely many jumps larger than 1 on all of $\mathbb{R}_+$, so the same argument as above applies to show that $\sum_{0<t} |\Delta f(t)|^p$ is finite. □

The following two results which will aid us in the proof of the Kunita-Watanabe inequality.

Lemma A.2.17. *Let $\alpha, \gamma \geq 0$ and $\beta \in \mathbb{R}$. It then holds that $|\beta| \leq \sqrt{\alpha}\sqrt{\gamma}$ if and only if $\lambda^2\alpha + 2\lambda\beta + \gamma \geq 0$ for all $\lambda \in \mathbb{Q}$.*

Proof. First note that by continuity, the requirement that $\lambda^2\alpha + 2\lambda\beta + \gamma \geq 0$ for all $\lambda \in \mathbb{Q}$ is equivalent to the same requirement for all $\lambda \in \mathbb{R}$.

Consider first the case $\alpha = 0$. If $|\beta| \le \sqrt{\alpha}\sqrt{\gamma}$, clearly $\beta = 0$, and the criterion is trivially satisfied. Conversely, assume that the criterion holds, which in this case is equivalent to $2\lambda\beta + \gamma \ge 0$ for all $\lambda \in \mathbb{Q}$. Letting λ tend to infinity or minus infinity depending on the sign of β, the requirement that γ be nonnegative forces $\beta = 0$, so that $\beta \le \sqrt{\alpha}\sqrt{\gamma}$. This proves the result in the case $\alpha = 0$. Next, consider the case $\alpha \neq 0$, so that $\alpha > 0$. The mapping $\lambda^2\alpha + 2\lambda\beta + \gamma \ge 0$ takes its minimum at $-\frac{\beta}{\alpha}$, and the minimum value is

$$\inf_{\lambda \in \mathbb{R}} \lambda^2\alpha + 2\lambda\beta + \gamma = \left(-\frac{\beta}{\alpha}\right)^2 \alpha - \frac{2\beta^2}{\alpha} + \gamma = \frac{1}{\alpha}(\alpha\gamma - \beta^2),$$

which is nonnegative if and only if $|\beta| \le \sqrt{\alpha}\sqrt{\gamma}$. This proves the result. □

Lemma A.2.18. *Let $f, g, h : \mathbb{R}_+ \to \mathbb{R}$ be in $\mathbf{FV}_0$, with f and g increasing. If it holds for all $0 \le s \le t$ that $|h(t) - h(s)| \le \sqrt{f(t) - f(s)}\sqrt{g(t) - g(s)}$, then for any measurable $x, y : \mathbb{R}_+ \to \mathbb{R}$, we have*

$$\int_0^\infty |x(t)y(t)||\,\mathrm{d}h(t)| \le \left(\int_0^\infty x(t)^2 \,\mathrm{d}f(t)\right)^{\frac{1}{2}} \left(\int_0^\infty y(t)^2 \,\mathrm{d}g(t)\right)^{\frac{1}{2}}.$$

Proof. Let μ_f, μ_g and μ_h be the measures corresponding to the finite variation mappings f, g and h. Clearly, the measures μ_f, μ_g and μ_h are all absolutely continuous with respect to $\nu = \mu_f + \mu_g + |\mu_h|$. Then, by the Radon-Nikodym Theorem, there exists densities φ_f, φ_g and φ_h of the three measures with respect to ν, and it therefore suffices to prove

$$\left(\int_0^\infty |x(t)y(t)||\varphi_h(t)| \,\mathrm{d}\nu(t)\right)^2 \le \left(\int_0^\infty x(t)^2\varphi_f(t) \,\mathrm{d}\nu(t)\right)\left(\int_0^\infty y(t)^2\varphi_g(t) \,\mathrm{d}\nu(t)\right).$$

To this end, we wish to argue that $|\varphi_h(t)| \le \sqrt{\varphi_f(t)}\sqrt{\varphi_g(t)}$, almost everywhere with respect to ν. By Lemma A.2.17, this is equivalent to proving that almost everywhere in t with respect to ν, it holds that for all $\lambda \in \mathbb{Q}$ that $\lambda^2\varphi_f(t) + 2\lambda\varphi_h(t) + \varphi_g(t) \ge 0$. As a countable intersection of null sets is again a null set, it suffices to prove that for any $\lambda \in \mathbb{Q}$, it holds that $\lambda^2\varphi_f(t) + 2\lambda\varphi_h(t) + \varphi_g(t) \ge 0$ almost everywhere with respect to ν. However, for any $0 \le s \le t$, we have

$$\int_s^t \lambda^2\varphi_f(t) + 2\lambda\varphi_h(t) + \varphi_g(t) \,\mathrm{d}\nu(t) = \lambda^2\mu_f(s,t] + 2\lambda\mu_h(s,t] + \mu_g(s,t],$$

and as $|\mu_h(s,t]| \le \sqrt{\mu_f(s,t]}\sqrt{\mu_g(s,t]}$ by assumption, the above is nonnegative by Lemma A.2.17. By an extension argument, we obtain that $\int_A \lambda^2\varphi_f(t) + 2\lambda\varphi_h(t) + \varphi_g(t) \,\mathrm{d}\nu(t) \ge 0$ for any $A \in \mathcal{B}_+$, in particular $\lambda^2\varphi_f(t) + 2\lambda\varphi_h(t) + \varphi_g(t) \ge 0$ almost everywhere with respect to ν. Thus, we finally conclude $|\varphi_h(t)| \le \sqrt{\varphi_f(t)}\sqrt{\varphi_g(t)}$. The Cauchy-Schwartz inequality then immediately yields

$$\begin{aligned}\int_0^\infty |x(t)y(t)||\varphi_h(t)|\,\mathrm{d}\nu(t) &\le \int_0^\infty |x(t)\sqrt{\varphi_f(t)}||y(t)\sqrt{\varphi_g(t)}|\,\mathrm{d}\nu(t)\\ &\le \left(\int_0^\infty x(t)^2\varphi_f(t)\,\mathrm{d}\nu(t)\right)^{\frac{1}{2}}\left(\int_0^\infty y(t)^2\varphi_g(t)\,\mathrm{d}\nu(t)\right)^{\frac{1}{2}}\\ &= \left(\int_0^\infty x(t)^2\,\mathrm{d}f(t)\right)^{\frac{1}{2}}\left(\int_0^\infty y(t)^2\,\mathrm{d}f(t)\right)^{\frac{1}{2}},\end{aligned}$$

as desired. □

Lemma A.2.19. *Let $f:\mathbb{R}_+\to\mathbb{R}$ be some càdlàg mapping. Let $U(f,a,b)$ denote the number of upcrossings from a to b of f, meaning that*

$$U(f,a,b)=\sup\{n \mid \exists\, 0\le s_1<t_1<\cdots s_n<t_n : f(s_k)<a, f(t_k)>b, k\le n\}.$$

It holds that $f(t)$ has a limit in $[-\infty,\infty]$ as t tends to infinity if $U(f,a,b)$ is finite for all $a,b\in\mathbb{Q}$ with $a<b$.

Proof. Assume that $U(f,a,b)$ is finite for all $a,b\in\mathbb{Q}$ with $a<b$. Assume, expecting a contradiction, that $f(t)$ does not converge to any limit in $[-\infty,\infty]$ as t tends to infinity. Then $\liminf_t f(t)<\limsup_t f(t)$. In particular, there exists $a,b\in\mathbb{Q}$ with $a<b$ such that $\liminf_t f(t)<a<b<\limsup_t f(t)$.

Now consider $U(f,a,b)$, we wish to derive a contradiction with our assumption that $U(f,a,b)$ is finite. If $U(f,a,b)$ is zero, either $f(t)\ge a$ for all $t\ge 0$, or $f(t)<a$ for some t and $f(t)\le b$ from a point onwards. In this first case, $\liminf_t f(t)\ge a$, and in the second case, $\limsup_t f(t)\le b$, both leading to contradictions. Therefore, $U(f,a,b)$ must be nonzero. As we have assumed that $U(f,a,b)$ is finite, we obtain that either $f(t)\ge a$ from a point onwards, or $f(t)\le b$ from a point onwards. In the first case, $\liminf_t f(t)\ge a$ and in the second case, $\limsup_t f(t)\le b$. Again, we obtain a contradiction, and so conclude that $f(t)$ must exist as a limit in $[-\infty,\infty]$. □

A.3 Convergence results and uniform integrability

In this section, we recall some basic results on convergence of random variables. Let $(\Omega,\mathcal{F},P)$ be a probability triple. Let X_n be a sequence of random variables and let X be another random variable. By $\mathcal{L}^p$, $p\ge 1$, we denote the variables X where $E|X|^p$ is finite. If $X_n(\omega)$

converges to $X(\omega)$ for all ω except on a null set, we say that X_n converges almost surely to X and write $X_n \xrightarrow{\text{a.s.}} X$. If it holds for all $\varepsilon > 0$ that $\lim_n P(|X_n - X| > \varepsilon) = 0$, we say that X_n converges in probability to X under P and write $X_n \xrightarrow{P} X$. If $\lim_n E|X_n - X|^p = 0$, we say that X_n converges to X in $\mathcal{L}^p$ and write $X_n \xrightarrow{\mathcal{L}^p} X$. Convergence in $\mathcal{L}^p$ and almost sure convergence both imply convergence in probability. Convergence in probability implies convergence almost surely along a subsequence.

The following lemmas will be useful at various points in the main text.

Lemma A.3.1. *Let X_n be a sequence of random variables, let X be another random variable and let $(F_n) \subseteq \mathcal{F}$. Assume that $X_n 1_{F_k} \xrightarrow{P} X 1_{F_k}$ for all $k \geq 1$ and that $\lim_k P(F_k^c) = 0$. Then $X_n \xrightarrow{P} X$ as well.*

Proof. For any $\varepsilon > 0$, we find

$$\begin{aligned} P(|X_n - X| > \varepsilon) &= P((|X_n - X| > \varepsilon) \cap F_k) + P((|X_n - X| > \varepsilon) \cap F_k^c) \\ &\leq P(|X_n 1_{F_k} - X 1_{F_k}| > \varepsilon) + P(F_k^c), \end{aligned}$$

and may therefore conclude $\limsup_n P(|X_n - X| > \varepsilon) \leq P(F_k^c)$. Letting k tend to infinity, we obtain $X_n \xrightarrow{P} X$. □

Lemma A.3.2. *Let (X_n) and (Y_n) be two sequences of variables convergent in probability to X and Y, respectively. If $X_n \leq Y_n$ almost surely for all n, then $X \leq Y$ almost surely.*

Proof. Picking nested subsequences, we find that for some subsequence, X_{n_k} tends almost surely to X and Y_{n_k} tends almost surely to Y. From the properties of ordinary convergence, we obtain $X \leq Y$ almost surely. □

Next, we consider the concept of uniform integrability, its basic properties and its relation to convergence of random variables. Let $(X_i)_{i \in I}$ be a family of random variables. We say that X_i is uniformly integrable if it holds that

$$\lim_{\lambda \to \infty} \sup_{i \in I} E|X_i| 1_{(|X_i| > \lambda)} = 0.$$

Note that as $\sup_{i \in I} E|X_i| 1_{(|X_i| > \lambda)}$ is decreasing in λ, the limit always exists in $[0, \infty]$. We will review some basic results about uniform integrability. We refer the results mainly for discrete sequences of variables, but many results extend to sequences indexed by $\mathbb{R}_+$ as well.

Lemma A.3.3. *Let $(X_i)_{i\in I}$ be some family of variables. (X_i) is uniformly integrable if and only if it holds that (X_i) is bounded in $\mathcal{L}^1$, and for every $\varepsilon > 0$, it holds that there is $\delta > 0$ such that whenever $F \in \mathcal{F}$ with $P(F) \leq \delta$, we have $E1_F|X_i| \leq \varepsilon$ for all $i \in I$.*

Proof. First assume that $(X_i)_{i\in I}$ is uniformly integrable. Clearly, we then have

$$\begin{aligned}\sup_{i\in I} E|X_i| &\leq \sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} + \sup_{i\in I} E|X_i|1_{(|X_i|\leq\lambda)}\\ &\leq \lambda + \sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)},\end{aligned}$$

and as the latter term converges to zero, it is in particular finite from a point onwards, and so $\sup_{i\in I} E|X_i|$ is finite, proving that (X_i) is bounded in $\mathcal{L}^1$. Now fix $\varepsilon > 0$. For any $\lambda > 0$, we have $E1_F|X_i| = E1_F|X_i|1_{(|X_i|>\lambda)} + E1_F|X_i|1_{(|X_i|\leq\lambda)} \leq \sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} + \lambda P(F)$. Therefore, picking λ so large that $\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} \leq \frac{\varepsilon}{2}$ and putting $\delta = \frac{\varepsilon}{2\lambda}$, we obtain $E1_F|X_i| \leq \varepsilon$ for all $i \in I$, as desired.

In order to obtain the converse, assume that $(X_i)_{i\in I}$ is bounded in $\mathcal{L}^1$ and that for all $\varepsilon > 0$, there is $\delta > 0$ such that whenever $F \in \mathcal{F}$ with $P(F) \leq \delta$, we have $E1_F|X_i| \leq \varepsilon$ for all $i \in I$. We need to prove that $(X_i)_{i\in I}$ is uniformly integrable. Fix $\varepsilon > 0$, we wish to prove that there is $\lambda > 0$ such that $\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} \leq \varepsilon$. To this end, let $\delta > 0$ be such that whenever $P(F) \leq \delta$, we have $E1_F|X_i| \leq \varepsilon$ for all $i \in I$. Note that by Markov's inequality, $P(|X_i| > \lambda) \leq \frac{1}{\lambda}E|X_i| \leq \frac{1}{\lambda}\sup_{i\in I} E|X_i|$, which is finite as $(X_i)_{i\in I}$ is bounded in $\mathcal{L}^1$. Therefore, there is $\lambda > 0$ such that $P(|X_i| > \lambda) \leq \delta$ for all i. For this λ, we then have $E|X_i|1_{(|X_i|>\lambda)} \leq \varepsilon$ for all $i \in I$, in particular $\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} \leq \varepsilon$ for this λ and all larger λ as well, proving $\lim_{\lambda\to\infty} \sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} = 0$ and thus proving uniform integrability. □

Lemma A.3.4. *The property of being uniformly integrable satisfies the following properties.*

1. *If $(X_i)_{i\in I}$ is a finite family of integrable variables, then (X_i) is uniformly integrable.*
2. *If $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ are uniformly integrable, then their union is uniformly integrable.*
3. *If $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are uniformly integrable, so is $(\alpha X_i + \beta Y_i)_{i\in I}$ for $\alpha, \beta \in \mathbb{R}$.*
4. *If $(X_i)_{i\in I}$ is uniformly integrable and $J \subseteq I$, then $(X_j)_{j\in J}$ is uniformly integrable.*
5. *If $(X_i)_{i\in I}$ is bounded in $\mathcal{L}^p$ for some $p > 1$, $(X_i)_{i\in I}$ is uniformly integrable.*
6. *If $(X_i)_{i\in I}$ is uniformly integrable and $|Y_i| \leq |X_i|$, then $(Y_i)_{i\in I}$ is uniformly integrable.*

Proof. **Proof of (1).** Assume that $(X_i)_{i\in I}$ is a finite family of integrable variables. The dominated convergence theorem then yields

$$\lim_{\lambda\to\infty}\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} \leq \lim_{\lambda\to\infty}\sum_{i\in I} E|X_i|1_{(|X_i|>\lambda)} = \sum_{i\in I} E\lim_{\lambda\to\infty}|X_i|1_{(|X_i|>\lambda)},$$

which is zero. Therefore, $(X_i)_{\in I}$ is uniformly integrable.

Proof of (2). As the maximum function is continuous, we find

$$\begin{aligned}
&\lim_{\lambda\to\infty}\max\{\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)}, \sup_{j\in J} E|Y_j|1_{(|Y_j|>\lambda)}\}\\
=\;&\max\{\lim_{\lambda\to\infty}\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)}, \lim_{\lambda\to\infty}\sup_{j\in J} E|Y_j|1_{(|Y_j|>\lambda)}\},
\end{aligned}$$

which is zero when the two families $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ are uniformly integrable, and the result follows.

Proof of (3). Assume that $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are uniformly integrable. If α and β are both zero, the result is trivial, so we assume that this is not the case. Let $\varepsilon > 0$. Using Lemma A.3.3, pick $\delta > 0$ such that whenever $P(F) \leq \delta$, we have the inequalities $E1_F|X_i| \leq \varepsilon(|\alpha|+|\beta|)^{-1}$ and $E1_F|Y_i| \leq \varepsilon(|\alpha|+|\beta|)^{-1}$ for any $i \in I$. Then

$$\begin{aligned}
E1_F|\alpha X_i + \beta Y_i| &\leq |\alpha|E1_F|X_i| + |\beta|E1_F|Y_i|\\
&\leq |\alpha|\varepsilon(|\alpha|+|\beta|)^{-1} + |\beta|\varepsilon(|\alpha|+|\beta|)^{-1} \leq \varepsilon,
\end{aligned}$$

so that by Lemma A.3.3, the result holds.

Proof of (4). As $J \subseteq I$, we have $\sup_{j\in J} E|X_j|1_{(|X_j|>\lambda)} \leq \sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)}$, and the result follows.

Proof of (5). Assume that $(X_i)_{i\in I}$ is bounded in $\mathcal{L}^p$ for some $p > 1$. We have

$$\lim_{\lambda\to\infty}\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} \leq \lim_{\lambda\to\infty}\lambda^{1-p}\sup_{i\in I} E|X_i|^p 1_{(|X_i|>\lambda)} \leq \sup_{i\in I} E|X_i|^p \lim_{\lambda\to\infty}\lambda^{1-p},$$

which is zero, as $p-1>0$, so $(X_i)_{i\in I}$ is uniformly integrable.

Proof of (6). In the case where $(X_i)_{\in I}$ is uniformly integrable and $(Y_i)_{i\in I}$ is such that $|Y_i| \leq |X_i|$, we get $E|Y_i|1_{(|Y_i|>\lambda)} \leq E|X_i|1_{(|X_i|>\lambda)}$ for all i, and it follows immediately that $(Y_i)_{i\in I}$ is uniformly integrable. □

Lemma A.3.5. *Let (X_n) be a sequence of random variables indexed by $\mathbb{N}$, and let X be another variable. X_n converges in $\mathcal{L}^1$ to X if and only if (X_n) is uniformly integrable and* *converges in probability to X. If (X_t) is a sequence of random variables indexed by $\mathbb{R}_+$, X_t converges to X in $\mathcal{L}^1$ if (X_t) is uniformly integrable and converges in probability to X.*

Proof. Consider first the discrete-time case. Assume that X_n converges to X in $\mathcal{L}^1$, we need to prove that (X_n) is uniformly integrable. We use the criterion from Lemma A.3.3. As (X_n) is convergent in $\mathcal{L}^1$, (X_n) is bounded in $\mathcal{L}^1$, and X_n converges to X in probability. Fix $\varepsilon > 0$ and let m be such that whenever $n \geq m$, $E|X_n - X| \leq \frac{\varepsilon}{3}$. As the finite-variable family $\{X_1, \ldots, X_m, X\}$ is uniformly integrable by Lemma A.3.4, using Lemma A.3.3 we may obtain $\delta > 0$ such that whenever $P(F) \leq \delta$, $E1_F|X| \leq \frac{\varepsilon}{3}$ and $E1_F|X_n| \leq \frac{\varepsilon}{3}$ for $n \leq m$. We then obtain that for all such $F \in \mathcal{F}$,

$$\begin{aligned}
\sup_n E1_F|X_n| &\leq \sup_{n\leq m} E1_F|X_n| + \sup_{n\geq m} E1_F|X_n| \\
&\leq \tfrac{\varepsilon}{3} + E1_F|X| + \sup_{n\geq m} E1_F|X_n - X| \\
&\leq \tfrac{2\varepsilon}{3} + \sup_{n\geq m} E|X_n - X| \leq \varepsilon,
\end{aligned}$$

so (X_n) is uniformly integrable.

Consider the converse statement, where we assume that (X_n) is uniformly integrable and converges to X in probability. As (X_n) is uniformly integrable, (X_n) is bounded in $\mathcal{L}^1$. Using that there is a subsequence (X_{n_k}) converging to X almost surely, we obtain by Fatou's lemma that $E|X| = E\lim_k |X_{n_k}| \leq \liminf_k E|X_{n_k}| \leq \sup_n E|X_n|$, so X is integrable. By Lemma A.3.4, $(X_n - X)$ is uniformly integrable. Let $\varepsilon > 0$. Using Lemma A.3.3, we pick $\delta > 0$ such that whenever $P(F) \leq \delta$, we have $E1_F|X_n - X| \leq \varepsilon$. As $X_n \xrightarrow{P} X$, there is m such that whenever $n \geq m$, we have $P(|X_n - X| > \varepsilon) \leq \delta$. For such n, we then find $E|X_n - X| = E1_{(|X_n - X| \leq \varepsilon)}|X_n - X| + E1_{(|X_n - X| > \varepsilon)}|X_n - X| \leq 2\varepsilon$, proving that X_n tends to X in $\mathcal{L}^1$.

As for the case of a family $(X_t)_{t\geq 0}$ indexed by $\mathbb{R}_+$, we see that the proof that X_t is convergent in $\mathcal{L}^1$ to X if X_t is convergent in probability to X and is uniformly integrable may be copied more or less verbatim from the discrete-time case. □

Lemma A.3.6. *Let X be any integrable random variable on probability space $(\Omega, \mathcal{F}, P)$. Let I be the set of all sub-σ-algebras of $\mathcal{F}$. Then, $(E(X|\mathcal{G}))_{\mathcal{G}\in I}$ is uniformly integrable.*

Proof. Using Jensen's inequality and the fact that $(E(|X||\mathcal{G}) > \lambda) \in \mathcal{G}$, we have

$$\sup_{\mathcal{G}\in I} E|E(X|\mathcal{G})|1_{(|E(X|\mathcal{G})|>\lambda)} \leq \sup_{\mathcal{G}\in I} EE(|X||\mathcal{G})1_{(E(|X||\mathcal{G})>\lambda)} = \sup_{\mathcal{G}\in I} E|X|1_{(E(|X||\mathcal{G})>\lambda)}.$$

Fix $\varepsilon > 0$, we show that for λ large enough, the above is smaller than ε. To this end, note that for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, we have $P(E(|X||\mathcal{G}) > \lambda) \leq \frac{1}{\lambda}EE(|X||\mathcal{G}) = \frac{1}{\lambda}E|X|$ by Markov's inequality. Applying Lemma A.3.3 with the family $\{X\}$, we know that there is $\delta > 0$ such that whenever $P(F) \leq \delta$, $E1_F|X| \leq \varepsilon$. Therefore, picking λ so large that $\frac{1}{\lambda}E|X| \leq \delta$, we obtain $P(E(|X||\mathcal{G}) > \lambda) \leq \delta$ and so $\sup_{\mathcal{G}\in I} E|X|1_{(E(|X||\mathcal{G})>\lambda)} \leq \varepsilon$. This concludes the proof. □

Lemma A.3.7 (Mazur's lemma). *Let (X_n) be sequence of variables bounded in $\mathcal{L}^2$. There exists a sequence (Y_n) such that each Y_n is a convex combination of a finite set of elements in $\{X_n, X_{n+1}, \ldots\}$ and (Y_n) is convergent in $\mathcal{L}^2$.*

Proof. Let α_n be the infimum of EZ^2, where Z ranges through all convex combinations of elements in $\{X_n, X_{n+1}, \ldots\}$, and define $\alpha = \sup_n \alpha_n$. If $Z = \sum_{k=n}^{K_n} \lambda_k X_k$ for some convex weights $\lambda_n, \ldots, \lambda_{K_n}$, we obtain $\sqrt{EZ^2} \leq \sum_{k=n}^{K_n} \lambda_k^n \sqrt{EX_k^2} \leq \sup_n \sqrt{EX_n^2}$, in particular we have $\alpha_n \leq \sup_n EX_n^2$ and so $\alpha \leq \sup_n EX_n^2$ as well, proving that α is finite. For each n, there is a variable Y_n which is a finite convex combination of elements in $\{X_n, X_{n+1}, \ldots\}$ such that $E(Y_n)^2 \leq \alpha_n + \frac{1}{n}$. Let n be so large that $\alpha_n \geq \alpha - \frac{1}{n}$, and let $m \geq n$, we then obtain

$$\begin{aligned} E(Y_n - Y_m)^2 &= 2EY_n^2 + 2EY_m^2 - E(Y_n + Y_m)^2 \\ &= 2EY_n^2 + 2EY_m^2 - 4E(\tfrac{1}{2}(Y_n + Y_m))^2 \\ &\leq 2(\alpha_n + \tfrac{1}{n}) + 2(\alpha_m + \tfrac{1}{m}) - 4\alpha_n \\ &= 2(\tfrac{1}{n} + \tfrac{1}{m}) + 2(\alpha_m - \alpha_n). \end{aligned}$$

As (α_n) is convergent, it is Cauchy. Therefore, the above shows that (Y_n) is Cauchy in $\mathcal{L}^2$, therefore convergent, proving the lemma. □

Lemma A.3.8. *Let (X_n) be a uniformly integrable sequence of variables. It then holds that*

$$\limsup_{n\to\infty} EX_n \leq E \limsup_{n\to\infty} X_n.$$

Proof. Since (X_n) is uniformly integrable, it holds that

$$0 \leq \lim_{\lambda\to\infty} \sup_n EX_n 1_{(X_n>\lambda)} \leq \lim_{\lambda\to\infty} \sup_n E|X_n|1_{(|X_n|>\lambda)} = 0$$

Let $\varepsilon > 0$ be given, we may then pick λ so large that $EX_n 1_{(X_n>\lambda)} \leq \varepsilon$ for all n. Now, the sequence $(\lambda - X_n 1_{(X_n \le \lambda)})_{n\ge 1}$ is nonnegative, and Fatou's lemma therefore yields

$$\begin{aligned}\lambda - E\limsup_{n\to\infty} X_n 1_{(X_n\le\lambda)} &= E\liminf_{n\to\infty}(\lambda - X_n 1_{(X_n\le\lambda)})\\ &\le \liminf_{n\to\infty} E(\lambda - X_n 1_{(X_n\le\lambda)})\\ &= \lambda - \limsup_{n\to\infty} EX_n 1_{(X_n\le\lambda)}.\end{aligned}$$

The terms involving the limes superior may be infinite and are therefore a priori not amenable to arbitrary arithmetic manipulation. However, by subtracting λ and multiplying by minus one, we may still conclude that $\limsup_{n\to\infty} EX_n 1_{(X_n\le\lambda)} \le E\limsup_{n\to\infty} X_n 1_{(X_n\le\lambda)}$. As we have ensured that $EX_n 1_{(X_n>\lambda)} \le \varepsilon$ for all n, this yields

$$\limsup_{n\to\infty} EX_n \le \varepsilon + E\limsup_{n\to\infty} X_n 1_{(X_n\le\lambda)} \le \varepsilon + E\limsup_{n\to\infty} X_n,$$

and as $\varepsilon > 0$ was arbitrary, the result follows. □

A.4 Discrete-time martingales

In this section, we review the basic results from discrete-time martingale theory. Assume given a probability field $(\Omega, \mathcal{F}, P)$. If $(\mathcal{F}_n)$ is a sequence of sub-σ-algebras of $\mathcal{F}$ indexed by $\mathbb{N}$ which are increasing in the sense that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, we say that $(\mathcal{F}_n)$ is a filtration. We then refer to $(\Omega, \mathcal{F}, (\mathcal{F}_n), P)$ as a filtered probability space. In the remainder of this section, we will assume given a filtered probability space of this kind.

A discrete-time stochastic process is a sequence $X = (X_n)$ of random variables defined on $(\Omega, \mathcal{F})$. If X_n is $\mathcal{F}_n$ measurable, we say that the process X is adapted. If X is adapted and $E(X_n|\mathcal{F}_k) = X_k$ whenever $n \ge k$, we say that X is a martingale. If instead $E(X_n|\mathcal{F}_k) \le X_k$, we say that X is a supermartingale and if $E(X_n|\mathcal{F}_k) \ge X_k$, we say that X is a submartingale. Any martingale is also a submartingale and a supermartingale. Furthermore, if X is a supermartingale, then $-X$ is a submartingale and vice versa.

A stopping time is a random variable $T : \Omega \to \mathbb{N} \cup \{\infty\}$ such that $(T \le n) \in \mathcal{F}_n$ for any $n \in \mathbb{N}$. We say that T is finite if T maps into $\mathbb{N}$. We say that T is bounded if T maps into a bounded subset of $\mathbb{N}$. If X is a stochastic process and T is a stopping time, we denote by X^T the process $X^T_n = X_{T\wedge n}$ and call X^T the process stopped at T. Furthermore, we define the stopping time σ-algebra $\mathcal{F}_T$ by putting $\mathcal{F}_T = \{A \in \mathcal{F} | A \cap (T \le n) \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}_0\}$. $\mathcal{F}_T$ is a σ-algebra, and if T is constant, the stopping time σ-algebra is the same as the filtration σ-algebra.

Lemma A.4.1 (Doob's upcrossing lemma)**.** *Let Z be a supermartingale which is bounded in $\mathcal{L}^1$. Define $U(Z,a,b) = \sup\{n \mid \exists\, 1 \leq s_1 < t_1 < \cdots s_n < t_n : Z_{s_k} < a, Z_{t_k} > b, k \leq n\}$ for any $a,b \in \mathbb{R}$ with $a < b$. We refer to $U(Z,a,b)$ as the number of upcrossings from a to b by Z. Then*

$$EU(Z,a,b) \leq \frac{|a| + \sup_n E|Z_n|}{b-a}$$

Proof. See Corollary II.48.4 of Rogers & Williams (2000a). □

Theorem A.4.2 (Doob's supermartingale convergence theorem)**.** *Let Z be a supermartingale. If Z is bounded in $\mathcal{L}^1$, Z is almost surely convergent. If Z is uniformly integrable, Z is also convergent in $\mathcal{L}^1$, and the limit Z_∞ satisfies that for all n, $E(Z_\infty|\mathcal{F}_n) \leq Z_n$ almost surely.*

Proof. That Z converges almost surely follows from Theorem II.49.1 of Rogers & Williams (2000a). The results for the case where Z is uniformly integrable follows from Theorem II.50.1 of Rogers & Williams (2000a). □

Theorem A.4.3 (Uniformly integrable martingale convergence theorem)**.** *Let M be a discrete-time martingale. The following are equivalent:*

1. *M is uniformly integrable.*
2. *M is convergent almost surely and in $\mathcal{L}^1$.*
3. *There is some integrable variable ξ such that $M_n = E(\xi|\mathcal{F}_n)$ for $n \geq 1$.*

In the affirmative, with M_∞ denoting the limit of M_n almost surely and in $\mathcal{L}^1$, we have for all $n \geq 1$ that $M_n = E(M_\infty|\mathcal{F}_n)$ almost surely, and $M_\infty = E(\xi|\mathcal{F}_\infty)$, where $\mathcal{F}_\infty = \sigma(\cup_{n=1}^\infty \mathcal{F}_n)$.

Proof. From Theorem II.50.1 in Rogers & Williams (2000a), it follows that if (1) holds, then (2) and (3) holds as well. From Theorem II.50.3 of Rogers & Williams (2000a), we find that if (3) holds, then (1) and (2) holds. Finally, (2) implies (1) by Lemma A.3.5.

In the affirmative case, Theorem II.50.3 of Rogers & Williams (2000a) shows that we have $M_\infty = E(\xi|\mathcal{F}_\infty)$, and so in particular, $M_n = E(\xi|\mathcal{F}_n) = E(E(\xi|\mathcal{F}_\infty)|\mathcal{F}_n) = E(M_\infty|\mathcal{F}_n)$ almost surely. □

Lemma A.4.4 (Doob's $\mathcal{L}^2$ inequality)**.** *Let M be a martingale such that $\sup_{n\geq 1} EM_n^2$ is finite. Then M is convergent almost surely and in $\mathcal{L}^2$ to a square-integrable variable M_∞, and $EM_\infty^{*2} \leq 4EM_\infty^2$, where $M_\infty^* = \sup_{n\geq 0}|M_n|$ and $M_\infty^{*2} = (M_\infty^*)^2$.*

Proof. This is Theorem II.52.6 of Rogers & Williams (2000a). □

Lemma A.4.5 (Optional sampling theorem)**.** *Let Z be a discrete-time supermartingale, and let $S \leq T$ be two stopping times. If Z is uniformly integrable, then Z is almost surely convergent, Z_S and Z_T are integrable, and $E(Z_T|\mathcal{F}_S) \leq Z_S$.*

Proof. That Z is almost surely convergent follows from Theorem II.49.1 of Rogers & Williams (2000a). That Z_S and Z_T are integrable and that $E(Z_T|\mathcal{F}_S) \leq Z_S$ then follows from Theorem II.59.1 of Rogers & Williams (2000a). □

Next, we consider backwards martingales. Let $(\mathcal{F}_n)_{n\geq 1}$ be a decreasing sequence of σ-algebras and let (Z_n) be some process. If Z_n is $\mathcal{F}_n$ measurable and integrable and $X_n = E(X_k|\mathcal{F}_n)$ for $n \geq k$, we say that (Z_n) is a backwards martingale. If instead $Z_n \leq E(Z_k|\mathcal{F}_n)$, we say that (Z_n) is a backwards supermartingale, and if $Z_n \geq E(Z_k|\mathcal{F}_n)$, we say that (Z_n) is a backwards submartingale.

Note that for both ordinary supermartingales and backwards supermartingales, the definition is essentially the same. A process Z is a supermartingale when, for $n \geq k$, $E(Z_n|\mathcal{F}_k) \leq Z_k$, while Z is a backwards supermartingale when, for $n \geq k$, $Z_n \leq E(Z_k|\mathcal{F}_n)$. Furthermore, if Z is a backwards supermartingale, then $-Z$ is a backwards submartingale and vice versa.

Theorem A.4.6 (Backwards supermartingale convergence theorem)**.** *Let $(\mathcal{F}_n)$ be a decreasing sequence of σ-algebras, and let (Z_n) be a backwards supermartingale. If $\sup_{n\geq 1} EZ_n$ is finite, then Z is uniformly integrable and convergent almost surely and in $\mathcal{L}^1$. Furthermore, the limit satisfies $Z_\infty \geq E(Z_n|\mathcal{F}_\infty)$, where $\mathcal{F}_\infty$ is the σ-algebra $\cap_{n=1}^\infty \mathcal{F}_n$.*

Proof. See Theorem II.51.1 of Rogers & Williams (2000a). □

Finally, we give a result on discrete-time compensators. Let $(A_n)_{n\geq 0}$ be an increasing adapted process with A_n integrable for each n. Assume that A has initial value zero, meaning that $A_0 = 0$. Define a process $(B_n)_{n\geq 0}$ by putting $B_n = \sum_{k=1}^n E(A_k - A_{k-1}|\mathcal{F}_{k-1})$. By Theorem II.54.1 of Rogers & Williams (2000a), $A - B$ is then a martingale. We refer to B as the compensator of A.

Lemma A.4.7. *Let A be an increasing adapted process such that A_∞ is bounded by c. Let B be the compensator of A. Then B_∞ is square-integrable and $EB_\infty^2 \le 2c^2$.*

Proof. First note that for $n \ge 1$,

$$\begin{aligned} B_n^2 &= 2B_n^2 - \sum_{k=0}^{n-1} B_{k+1}^2 - B_k^2 = 2\sum_{k=0}^{n-1} B_n(B_{k+1} - B_k) - \sum_{k=0}^{n-1} B_{k+1}^2 - B_k^2 \\ &= \sum_{k=0}^{n-1} 2(B_n - B_k)(B_{k+1} - B_k) - (B_{k+1} - B_k)^2. \end{aligned}$$

Taking means and recalling that B_{k+1} is $\mathcal{F}_k$ measurable, this yields

$$\begin{aligned} EB_n^2 &\le \sum_{k=0}^{n-1} 2E(B_n - B_k)(B_{k+1} - B_k) = E\sum_{k=0}^{n-1} 2E(B_n - B_k|\mathcal{F}_k)(B_{k+1} - B_k) \\ &= E\sum_{k=0}^{n-1} 2E(A_n - A_k|\mathcal{F}_k)(B_{k+1} - B_k) \le 2cE\sum_{k=0}^{n-1} B_{k+1} - B_k \\ &= 2cEB_n = 2cEA_n \le 2c^2. \end{aligned}$$

Letting n tend to infinity, the monotone convergence theorem yields that B_∞^2 is square-integrable and that $EB_\infty^2 = E\lim_n B_n^2 = \lim_n EB_n^2 \le 2c^2$, as desired. □

Appendix B

Hints for exercises

B.1 Hints for Chapter 1

Hints for exercise 1.5.1. Pick n so that $n \leq t < n+1$ and ε so small that $t + \varepsilon < n + 1$. Rewrite $\cap_{s>t}\mathcal{F}_s$ in terms of $\mathcal{F}_s$ for $t < s \leq t + \varepsilon$. ∘

Hints for exercise 1.5.2. First pick $t > 0$ and show that the restriction of X to $[0,t) \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable. ∘

Hints for exercise 1.5.3. First argue that

$$(T < t) = \cup_{n=1}^{\infty}(\ \exists\ s \in \mathbb{R}_+ : s \leq t - \tfrac{1}{n} \text{ and } X_s \in F).$$

Next, show that

$$(\ \exists\ s \in \mathbb{R}_+ : s \leq t - \tfrac{1}{n} \text{ and } X_s \in F) = \cap_{k=1}^{\infty}(\ \exists\ q \in \mathbb{Q}_+ : q \leq t - \tfrac{1}{n} \text{ and } X_q \in U_k),$$

where $U_k = \{x \in \mathbb{R} \mid \exists y \in F : |x - y| < \frac{1}{k}\}$. Argue that the right-hand side is in $\mathcal{F}_t$ and use this to obtain the result. ∘

Hints for exercise 1.5.4. First argue that it suffices to show that for all $t > 0$, the set

$$(\ \exists\ s \in \mathbb{R}_+ : s \leq t \text{ and it holds that } X_s \in F \text{ or } X_{s-} \in F)$$

is $\mathcal{F}_t$ measurable. To do so, define $U_k = \{x \in \mathbb{R} \mid \exists y \in F : |x-y| < \frac{1}{k}\}$ and show that

$$\begin{aligned} & (\ \exists\ s \in \mathbb{R}_+ : s \leq t \text{ and it holds that } X_s \in F \text{ or } X_{s-} \in F) \\ = \quad & \cap_{k=1}^\infty (\ \exists\ q \in \mathbb{Q}_+ \cup \{t\} : q \leq t \text{ and it holds that } X_q \in U_k). \end{aligned}$$

$\circ$

Hints for exercise 1.5.5. Use Exercise 1.5.3 to conclude that T is a stopping time. To show that $X_T = a$ whenever $T < \infty$, show that whenever T is finite, there is a sequence (u_n) depending on ω such that $T \leq u_n \leq T + \frac{1}{n}$ and such that $X_{u_n} = a$. Use this to obtain the desired result. $\circ$

Hints for exercise 1.5.6. For the inclusion $\mathcal{F}_{S \vee T} \subseteq \sigma(\mathcal{F}_S, \mathcal{F}_T)$, use that for any F, it holds that $F = (F \cap (S \leq T)) \cup (F \cap (T \leq S))$. $\circ$

Hints for exercise 1.5.7. Use Lemma 1.1.9, to show $\mathcal{F}_T \subseteq \cap_{n=1}^\infty \mathcal{F}_{T_n}$. In order to obtain the other inclusion, let $F \in \cap_{n=1}^\infty \mathcal{F}_{T_n}$. Show that $F \cap (T \leq t) = \cap_{n=1}^\infty \cup_{m=1}^\infty \cap_{k=m}^\infty F \cap (T_k \leq t + \frac{1}{n})$. Use this and the right-continuity of the filtration to prove that $F \cap (T \leq t) \in \mathcal{F}_t$, and conclude $\cap_{n=1}^\infty \mathcal{F}_{T_n} \subseteq \mathcal{F}_T$ from this. $\circ$

Hints for exercise 1.5.8. To show that M is not uniformly integrable, assume that M is in fact uniformly integrable. Prove that $\frac{1}{t} M_t$ then converges to zero in $\mathcal{L}^1$ and obtain a contradiction from this. $\circ$

Hints for exercise 1.5.10. In order to prove that M^α is a martingale, recall that for any $0 \leq s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and has a normal distribution with mean zero and variance $t - s$. In order to obtain the result on the Laplace transform of the stopping time T, first reduce to the case of $a \geq 0$. Note that by the properties of Brownian motion, T is almost surely finite. Show that $(M^\alpha)^T$ is uniformly integrable and use the optional sampling theorem in order to obtain the result. $\circ$

Hints for exercise 1.5.11. For the first process, use $W_t^3 = (W_t - W_s)^3 + 3W_t^2 W_s - 3W_t W_s^2 + W_s^3$ and calculate conditional expectations using the properties of the $\mathcal{F}_t$ Brownian motion. Apply a similar method to obtain the martingale property of the second process. $\circ$

Hints for exercise 1.5.12. To show the equality for $P(T < \infty)$, consider the martingale M^α defined in Exercise 1.5.10 by $M_t^\alpha = \exp(\alpha W_t - \frac{1}{2}\alpha^2 t)$ for $\alpha \in \mathbb{R}$. Show that the equality $E1_{(T<\infty)} M_T^{2b} = \exp(2ab) P(T < \infty)$ holds. Recalling that $\lim_{t \to \infty} \frac{W_t}{t} = 0$, use the optional sampling theorem and the dominated convergence theorem to show that $E1_{(T<\infty)} M_T^{2b} = 1$.

Use this to obtain the result. ∘

Hints for exercise 1.5.13. First show that we always have $0 < T \le 1$ and $W_T^2 = a(1-T)$. From Theorem 1.2.13 and Exercise 1.5.11, it is known that the processes $W_t^2 - t$ and $W_t^4 - 6tW_t^2 + 3t^2$ are in $\mathbf{c}\mathcal{M}$. Use these facts to obtain expressions for ET and ET^2. ∘

Hints for exercise 1.5.14. Use that $N_t^2 = (N_t - N_s)^2 + 2N_tN_s - N_s^2$ and that $N_t - N_s$ is Poisson distributed with paramter $t - s$ and independent of $\mathcal{F}_s$. ∘

Hints for exercise 1.5.15. Use that for $0 \le s \le t$, $N_t - N_s$ is independent of $\mathcal{F}_s$, and recall the formula for the moment generating function of the Poisson distribution. ∘

B.2 Hints for Chapter 2

Hints for exercise 2.4.2. Define $U_n = (a - 1/n, a + 1/n)$, $S_n = \inf\{t \ge 0 \mid X_t \in U_n\}$ and $T_n = S_n \wedge n$. Argue that (T_n) is an announcing sequence for T. ∘

Hints for exercise 2.4.3. Apply Lemma 2.2.4. ∘

Hints for exercise 2.4.5. Let (q_n) be an enumeration of q_n, define a sequence of stopping times (T_n) by putting $T_n = n \wedge S_{(S+q_n \ge T)} \wedge (S + q_n)_{(S+q_n<T)}$ and apply Exercise 2.4.4. ∘

Hints for exercise 2.4.6. Apply Lemma 2.2.6 and Lemma 2.2.5. ∘

Hints for exercise 2.4.7. Apply Lemma 2.2.7. ∘

Hints for exercise 2.4.9. To show that T is predictable when $(T = S) \in \mathcal{F}_{S-}$ for all predictable stopping times S, take a sequence of predictable times (S_n) with the property that $[\![T]\!] \subseteq \cup_{n=1}^\infty [\![S_n]\!]$, consider $T_n = (S_n)_{(T=S_n)}$ and apply Lemma 2.1.9. ∘

Hints for exercise 2.4.11. Define a sequence of stopping times (T_n) by approximating T through the dyadic rationals $\mathbb{D}_+$ from above. ∘

B.3 Hints for Chapter 3

Hints for exercise 3.5.2. In order to obtain that $M \in \mathcal{M}^u$ when $(M_T)_{T \in \mathcal{C}}$ is uniformly integrable, use Lemma 1.2.8 and Lemma A.3.5. ◦

Hints for exercise 3.5.3. Apply Fatou's lemma. ◦

Hints for exercise 3.5.4. First consider the case where $M \in \mathcal{M}^u$ and note that in this case, $|M_T|$ is integrable for all stopping times T. ◦

Hints for exercise 3.5.5. First consider $M \in \mathcal{M}^u$ and apply Lemma 3.1.8. ◦

Hints for exercise 3.5.6. Apply Exercise 3.5.5. ◦

Hints for exercise 3.5.9. First consider the case where $A \in \mathcal{A}$. Use Lemma 3.2.6 to obtain $A \in \mathcal{A}_\ell^i$. For the case where $A \in \mathcal{V}$, decompose A as in Lemma 1.4.1 and use Theorem 2.3.9 to show that each component in the decomposition is predictable. ◦

Hints for exercise 3.5.10. To show that the process $\int_0^t N_{s-} \,\mathrm{d}M_s$ is in $\mathcal{M}_\ell$, apply Lemma 3.3.2. To show that the process $\int_0^t N_s \,\mathrm{d}M_s$ is not in $\mathcal{M}_\ell$, assume to the contrary that the process is a local martingale. Let T_n be the n'th jump time of N. Argue that both $\int_0^t N_{s-} \,\mathrm{d}M_s$ and $\int_0^t N_s \,\mathrm{d}M_s$, when stopped at T_n, are in $\mathcal{M}^u$. Obtain a contradiction from this. ◦

Hints for exercise 3.5.11. Fix $\lambda > 0$. Use Lemma 3.3.2 and its proof to obtain

$$E \exp(-\lambda \Pi_p^* A_T) = E \int_0^\infty \exp(-\lambda \Pi_p^* A_s) \,\mathrm{d}\Pi_p^* A_s.$$

Apply Lemma A.2.14 to calculate the right-hand side in the above. ◦

Hints for exercise 3.5.13. In order to obtain the final two convergences in probability, apply the relation $W_t^2 = 2\sum_{k=1}^{2^n} W_{t_{k-1}^n}(W_{t_k^n} - W_{t_{k-1}^n}) + \sum_{k=1}^{2^n}(W_{t_k^n} - W_{t_{k-1}^n})^2$. ◦

Hints for exercise 3.5.14. Apply Theorem 3.3.10. ◦

Hints for exercise 3.5.17. Apply Lemma 3.4.4. ◦

Hints for exercise 3.5.18. Apply Lemma 3.4.6. ◦

B.4 Hints for Chapter 4

Hints for exercise 4.4.1. Apply Theorem 3.1.9. ◦

Hints for exercise 4.4.2. Apply Lemma 4.2.3 and Exercise 3.5.4. ◦

Hints for exercise 4.4.5. To prove that $X = X_0 + M + A$ with $M \in \mathcal{M}_\ell$ almost surely continuous and $A \in \mathcal{V}$, apply Theorem 3.4.7 Theorem 3.4.11 to obtain the desired decomposition. ◦

Hints for exercise 4.4.6. Use Theorem 3.4.11. ◦

Hints for exercise 4.4.7. First consider the case where $M \in \mathcal{M}^u$, and apply Lemma 3.1.8 and Lemma 3.2.10 to obtain that $M^{T-} \in \mathcal{M}_\ell$ in this case. ◦

Hints for exercise 4.4.12. Prove that $\frac{1}{t}(H \cdot M)_t$ and $\frac{N_t}{t}$ converge in probability to zero and one, respectively. Use this to obtain the result. ◦

Hints for exercise 4.4.13. To show that $[M, A] \in \mathcal{M}_\ell$, first use Theorem 3.3.1 to argue that it suffices to consider the two cases $M \in \mathcal{M}_\ell^b$ and $M \in \mathbf{fv}\mathcal{M}_\ell$. In order to prove the result for $M \in \mathcal{M}_\ell^b$, first consider the case $M \in \mathcal{M}^b$ and $A \in \mathcal{A}^i$ with A predictable. Argue that in this case, ΔM is almost surely integrable on $[0, \infty)$ with respect to A, and the result has finite mean, in particular implying that $[M, A] \in \mathcal{V}^i$. Defining $T_t = \inf\{s \geq 0 | A_s \geq t\}$, argue that T_t is a predictable stopping time and use Lemma 3.1.8 and Lemma A.2.14 to obtain $E[M, A]_\infty = 0$. Apply Lemma 1.2.8 to obtain that $[M, A] \in \mathcal{M}^u$ in this case.

Extend the result to $M \in \mathcal{M}_\ell^b$ and $A \in \mathcal{V}$ by localisation arguments and Lemma 4.2.3. ◦

Hints for exercise 4.4.14. Apply Exercise 4.4.13. ◦

Hints for exercise 4.4.15. Use Theorem 4.2.9 to argue that it suffices to show that $(\Delta H \cdot M)_t = \sum_{0<s\leq t} \Delta H_s \Delta X_s$. Use Exercise 4.4.4 to argue that $\Delta H = \Delta B$ for a predictable process $B \in \mathcal{V}$. Use Exercise 4.4.13 to obtain the desired result. ◦

Hints for exercise 4.4.16. First show that it suffices to prove the convergence to zero in probability of the variables that $(W_{t+h} - W_t)^{-1} \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty [\![} \, \mathrm{d}W_s$, where the indicator $1_{[\![t,\infty [\![}$ is included to ensure that the integrand is progressively measurable. To show this, fix

$\delta > 0$ and show that

$$\frac{1_{(|W_{t+h}-W_t|>\delta)}}{W_{t+h}-W_t}\int_t^{t+h}(H_s-H_t)1_{[\![t,\infty[\![}\,\mathrm{d}W_s \xrightarrow{P} 0,$$

using Chebychev's inequality and the results from Lemma 4.4.10. Then apply Lemma A.3.1 to obtain the result. ○

Hints for exercise 4.4.18. To prove that X has the same distribution as a Brownian motion for $H = \frac{1}{2}$, let W be a Brownian motion and show that X and W have the same finite-dimensional distributions.

To show that X is not in $\mathbf{c}\mathcal{S}$ when $H \neq \frac{1}{2}$, first fix $t \geq 0$, put $t_k^n = tk2^{-n}$ and consider $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|^p$ for $p \geq 0$. Use the fact that normal distributions are determined by their mean and covariance structure to argue that the distribution of $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|^p$ is the same as the distribution of $2^{-npH}\sum_{k=1}^{2^n}|X_k - X_{k-1}|^p$. Show that the process $(X_k - X_{k-1})_{k\geq 1}$ is stationary. Now recall that the ergodic theorem for discrete-time stationary processes states that for a stationary process $(Y_n)_{n\geq 1}$ and a mapping $f : \mathbb{R} \to \mathbb{R}$ such that $f(Y_1)$ is integrable, it holds that $\frac{1}{n}\sum_{k=1}^n f(Y_k)$ converges almost surely and in $\mathcal{L}^1$. Use this to argue that $\frac{1}{2^n}\sum_{k=1}^{2^n}|X_k - X_{k-1}|^p$ converges almost surely and in $\mathcal{L}^1$ to a variable Z_p which is not almost surely zero.

Finally, use this to prove that X is not in $\mathbf{c}\mathcal{S}$ when $H \neq \frac{1}{2}$. To do so, first consider the case $H < \frac{1}{2}$. In this case, assume that $X \in \mathbf{c}\mathcal{S}$ and seek a contradiction. Use the result of Exercise 4.4.17 to show that $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|^{\frac{1}{H}}$ converges to zero in probability. Obtain a contradiction with the results obtained above. In the case where $H > \frac{1}{2}$, use that $\frac{1}{H} < 2$ and Exercise 4.4.17 to show that $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|^2$ converges in probability to zero, and use this to argue that $[X]$ is evanescent. Conclude that X has paths of finite variation, and use this to obtain a contradiction with our earlier results. ○

Hints for exercise 4.4.20. Apply Itô's formula to the two-dimensional continuous semimartingale (t, W_t). ○

Hints for exercise 4.4.21. Use Theorem 4.3.2 to show that $\int_0^t f(s)\,\mathrm{d}W_s$ is the limit in probability of a sequence of variables whose distribution may be calculated explicitly. Use that convergence in probability implies weak convergence to obtain the desired result. ○

Hints for exercise 4.4.23. In the case where $i = j$, use Itô's formula with the function $f(x) = x^2$, and in the case $i \neq j$, use Itô's formula with the function $f(x, y) = xy$. Afterwards, apply Lemma 4.1.13 and Lemma 4.2.11 to obtain the result. ○

Hints for exercise 4.4.24. First note that in order to prove the result, it suffices to prove that the sequence $\sum_{k=1}^{2^n}(M_{t_k} - M_{t_{k-1}})^2$ is bounded in $\mathcal{L}^2$. To do so, write

$$E\left(\sum_{k=1}^{2^n}(M_{t_k} - M_{t_{k-1}})^2\right)^2 = E\sum_{k=1}^{2^n}(M_{t_k} - M_{t_{k-1}})^4 + E\sum_{k\neq i}(M_{t_k} - M_{t_{k-1}})^2(M_{t_i} - M_{t_{i-1}})^2.$$

Let $C \geq 0$ be a constant such that $|M_t| \leq C$ for all $t \geq 0$. By repeated use of the martingale property, prove that the first term is less than $4C^2$ and that the second term is less than $2C^2$, thus proving boundedness in $\mathcal{L}^2$. ∘

Appendix C

Solutions for exercises

C.1 Solutions for Chapter 1

Solution to exercise 1.5.1. Fix $t \geq 0$ and let $n = [t]$, the largest integer smaller than or equal to t. We then have $n \leq t < n+1$, and so $\mathcal{F}_t = \mathcal{G}_n$. Fix $\varepsilon > 0$ so small that $t + \varepsilon < n+1$, we then also have $\mathcal{F}_s = \mathcal{G}_n$ for $t < s \leq t + \varepsilon$, and it follows that

$$\cap_{s>t}\mathcal{F}_s = \cap_{t<s\leq t+\varepsilon}\mathcal{F}_s = \mathcal{G}_n = \mathcal{F}_t,$$

as desired. □

Solution to exercise 1.5.2. Fix $t > 0$ and $0 < \delta < t$. By our assumptions, it holds for any $\varepsilon > 0$ that the restriction $X_{|[0,t-\delta]\times\Omega}$ of X to $[0, t-\delta]\times\Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_{t-\delta+\varepsilon}$ measurable. Picking ε smaller than δ, we obtain that $X_{|[0,t-\delta]\times\Omega}$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable. This means that for any $B \in \mathcal{B}$ and $\delta > 0$, it holds that

$$\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid X_t(\omega) \in B\} \cap [0, t-\delta] \times \Omega \in \mathcal{B}_t \otimes \mathcal{F}_t \cap [0, t-\delta] \times \Omega.$$

As $\mathcal{B}_t \otimes \mathcal{F}_t \cap [0, t-\delta] \times \Omega \subseteq \mathcal{B}_t \otimes \mathcal{F}_t$, we also obtain that for all $B \in \mathcal{B}$ and $\delta > 0$,

$$\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid X_t(\omega) \in B\} \cap [0, t-\delta] \times \Omega \in \mathcal{B}_t \otimes \mathcal{F}_t.$$

As a consequence, we find that

$$\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid X_t(\omega) \in B\} \cap [0, t) \times \Omega \in \mathcal{B}_t \otimes \mathcal{F}_t.$$

Since we have assumed that X is adapted, it holds that X_t is $\mathcal{F}_t$ measurable, and the above therefore shows that the restriction of X to $[0,t]\times\Omega$ is $\mathcal{B}_t\otimes\mathcal{F}_t$ measurable, so X is progressive with respect to $(\mathcal{F}_t)_{t\geq 0}$. □

Solution to exercise 1.5.3. Define $U_n = \{x\in\mathbb{R} \mid \exists y\in F : |x-y| < \frac{1}{n}\}$. We claim that U_n is open, that U_n decreases to F and that $\overline{U}_{n+1}\subseteq U_n$, where $\overline{U}_{n+1}$ denotes the closure of U_{n+1}. As we have that $U_n = \cup_{y\in F}\{x\in\mathbb{R} \mid |x-y| < \frac{1}{n}\}$, U_n is open as a union of open sets. We have $F\subseteq U_n$ for all n, and conversely, if $x\in\cap_{n=1}^\infty U_n$, we have that there is a sequence (y_n) in F such that $|y_n - x| \leq \frac{1}{n}$ for all n. In particular, y_n tends to x, and as F is closed, we conclude $x\in F$. Thus, $F = \cap_{n=1}^\infty U_n$. Furthermore, if x is in the closure $\overline{U}_{n+1}$, there is a sequence (x_k) in U_{n+1} such that $|x_k - x| \leq \frac{1}{k}$, and there is a sequence (y_k) in F such that $|x_k - y_k| < \frac{1}{n+1}$, showing that $|x - y_k| < \frac{1}{k} + \frac{1}{n+1}$. Taking k so large that $\frac{1}{k}+\frac{1}{n+1}\leq\frac{1}{n}$, we see that $\overline{U}_{n+1}\subseteq U_n$.

Now note that whenever $t>0$, we have

$$(T<t) = (\,\exists\, s\in\mathbb{R}_+ : s<t \text{ and } X_s\in F) = \cup_{n=1}^\infty(\,\exists\, s\in\mathbb{R}_+ : s\leq t-\tfrac{1}{n} \text{ and } X_s\in F),$$

so by Lemma 1.1.9, it suffices to prove that $(\,\exists\, s\in\mathbb{R}_+ : s\leq t \text{ and } X_s\in F)$ is $\mathcal{F}_t$ measurable for all $t>0$. We claim that

$$(\,\exists\, s\in\mathbb{R}_+ : s\leq t \text{ and } X_s\in F) = \cap_{k=1}^\infty(\,\exists\, q\in\mathbb{Q}_+ : q\leq t \text{ and } X_q\in U_k).$$

To see this, first consider the inclusion towards the right. If there is $s\in\mathbb{R}_+$ with $s\leq t$ and $X_s\in F$, then we also have $X_s\in U_k$ for all k. As U_k is open, there is $\varepsilon>0$ such that the ball of size ε around X_s is in U_k. In particular, there is $q\in\mathbb{Q}_+$ with $q\leq t$ such that $X_q\in U_k$. This proves the inclusion towards the right. In order to obtain the inclusion towards the left, assume that for all k, there is $q_k\in\mathbb{Q}_+$ with $q_k\leq t$ such that $X_{q_k}\in U_k$. As $[0,t]$ is compact, there exists $s\in\mathbb{R}_+$ with $s\leq t$ such that for some subsequence, $\lim_m q_{k_m} = s$. By continuity, $X_s = \lim_m X_{q_{k_m}}$. As $X_{q_{k_m}}\in U_{k_m}$, we have for any i that $X_{q_{k_m}}\in U_i$ for m large enough. Therefore, we conclude $X_s\in\cap_{i=1}^\infty\overline{U}_i\subseteq\cap_{i=1}^\infty U_i = F$, proving the other inclusion. This shows the desired result. □

Solution to exercise 1.5.4. First note that whenever $t>0$, it holds that

$$\begin{aligned}(T<t) &= (\,\exists\, s\in\mathbb{R}_+ : s<t \text{ and it holds that } X_s\in F \text{ or } X_{s-}\in F)\\ &= \cup_{n=1}^\infty(\,\exists\, s\in\mathbb{R}_+ : s\leq t-\tfrac{1}{n} \text{ and it holds that } X_s\in F \text{ or } X_{s-}\in F).\end{aligned}$$

Therefore, applying Lemma 1.1.9, it suffices to prove that for all $t>0$, the set

$$(\,\exists\, s\in\mathbb{R}_+ : s\leq t \text{ and it holds that } X_s\in F \text{ or } X_{s-}\in F)$$

is $\mathcal{F}_t$ measurable. To this end, fix $t > 0$. Define $U_n = \{x \in \mathbb{R} \mid \exists y \in F : |x - y| < \frac{1}{n}\}$. As in 1.5.3, we find that each U_n is open, the sequence (U_n) decreases to F and $\overline{U}_{n+1} \subseteq U_n$. We claim that

$$\begin{aligned} & (\ \exists\, s \in \mathbb{R}_+ : s \leq t \text{ and it holds that } X_s \in F \text{ or } X_{s-} \in F) \\ = \ & \cap_{k=1}^{\infty}(\ \exists\, q \in \mathbb{Q}_+ \cup \{t\} : q \leq t \text{ and it holds that } X_q \in U_k). \end{aligned}$$

To prove this, define

$$\begin{aligned} C &= (\ \exists\, s \in \mathbb{R}_+ : s \leq t \text{ and it holds that } X_s \in F \text{ or } X_{s-} \in F) \\ D &= \cap_{k=1}^{\infty}(\ \exists\, q \in \mathbb{Q}_+ \cup \{t\} : q \leq t \text{ and it holds that } X_q \in U_k). \end{aligned}$$

First assume that $\omega \in C$, meaning that there exists $s \in \mathbb{R}_+$ with $s \leq t$ such that $X_s(\omega) \in F$. It $s = t$, $\omega \in D$ is immediate. Therefore, assume $s < t$. We then also have $X_s(\omega) \in U_k$ for all $k \geq 1$, and as X is right-continuous and U_k is open, there is for each $k \geq 1$ a $q \in \mathbb{Q}_+$ with $0 \leq q < t$ such that $X_q(\omega) \in U_k$, again yielding $\omega \in D$. Assume next that there exists $s \in \mathbb{R}_+$ with $s \leq t$ such that $X_{s-}(\omega) \in F$. We then also obtain $X_{s-}(\omega) \in U_k$ for all $k \geq 1$. If $s = 0$, we have $s \in \mathbb{Q}_+$ and $X_{s-} = X_s$, so $\omega \in D$. If $s > 0$, there exists $q \in \mathbb{Q}_+$ with $0 \leq q < s \leq t$ such that $X_s(\omega) \in U_k$, so $\omega \in D$. All in all, this shows $C \subseteq D$.

Next, we consider the other inclusion. Assume that $\omega \in D$. For each k, let q_k be an element of $\mathbb{Q}_+ \cup \{t\}$ with $q_k \leq t$ such that $X_{q_k}(\omega) \in U_k$. The sequence (q_k) has a monotone subsequence (q_{k_i}) with limit $s \in [0, t]$. If q_{k_i} is decreasing, we find that $\lim_i X_{q_{k_i}} = X_s(\omega)$. If q_{k_i} is increasing with $q_{k_i} < s$ infinitely often, we obtain $q_{k_i} < s$ eventually and so we conclude $\lim_i X_{q_{k_i}}(\omega) = X_{s-}(\omega)$. If q_{k_i} is increasing with $q_{k_i} \geq s$ eventually, we have $\lim_i X_{q_{k_i}}(\omega) = X_s(\omega)$. In all cases, the sequence $X_{q_{k_i}}(\omega)$ is convergent to either $X_s(\omega)$ or $X_{s-}(\omega)$. As $X_{q_{k_i}}(\omega) \in U_{k_i}$, we also obtain the the limit is in $\cap_{i=1}^{\infty}\overline{U}_i$, which is a subset of F. Thus, we either have $X_s(\omega) \in F$ or $X_{s-}(\omega) \in F$, so $\omega \in C$ and thus $D \subseteq C$.

This proves $C = D$. As $D \in \mathcal{F}_t$ since X is adapted, T is a stopping time. □

Solution to exercise 1.5.5. As one-point sets are closed, we know from Exercise 1.5.3 that T is a stopping time. When $T < \infty$, it holds that $\{t \geq 0 | X_t = a\}$ is nonempty, and for any n, $T + \frac{1}{n}$ is not a lower bound for the set $\{t \geq 0 \mid X_t = a\}$. Therefore, there is $u_n < T + \frac{1}{n}$ such that $u \in \{t \geq 0 \mid X_t = a\}$. Furthermore, as T is a lower bound for $\{t \geq 0 \mid X_t = a\}$, $u_n \geq T$. Thus, by continuity, $X_T = \lim_n X_{u_n} = a$. □

Solution to exercise 1.5.6. By Lemma 1.1.9, $\mathcal{F}_S \subseteq \mathcal{F}_{S \vee T}$ and $\mathcal{F}_T \subseteq \mathcal{F}_{S \vee T}$, showing that

$\sigma(\mathcal{F}_S, \mathcal{F}_T) \subseteq \mathcal{F}_{S \vee T}$. We need to prove the other implication. Let $F \in \mathcal{F}_{S \vee T}$. We find

$$\begin{aligned} F \cap (S \leq T) \cap (T \leq t) &= F \cap (S \leq T) \cap (S \vee T \leq t) \\ &= (F \cap (S \vee T \leq t)) \cap ((S \leq T) \cap (S \vee T \leq t)). \end{aligned}$$

Here, $F \cap (S \vee T \leq t) \in \mathcal{F}_t$ as $F \in \mathcal{F}_{S \vee T}$, and $(S \leq T) \cap (S \vee T \leq t) \in \mathcal{F}_t$ as we have $(S \leq T) \in \mathcal{F}_{S \wedge T} \subseteq \mathcal{F}_{S \vee T}$ by Lemma 1.1.11. We conclude $F \cap (S \leq T) \in \mathcal{F}_T$. Analogously, $F \cap (T \leq S) \in \mathcal{F}_S$. From this, we obtain $F \in \sigma(\mathcal{F}_S, \mathcal{F}_T)$, as desired. □

Solution to exercise 1.5.7. First note that by Lemma 1.1.10, T is a stopping time. In particular, $\mathcal{F}_T$ is well-defined. Using Lemma 1.1.9, the relation $T \leq T_n$ yields $\mathcal{F}_T \subseteq \mathcal{F}_{T_n}$, so that $\mathcal{F}_T \subseteq \cap_{n=1}^\infty \mathcal{F}_{T_n}$. Conversely, let $F \in \cap_{n=1}^\infty \mathcal{F}_{T_n}$, we want to show $F \in \mathcal{F}_T$, and to this end, we have to show $F \cap (T \leq t) \in \mathcal{F}_t$ for any $t \geq 0$. Fixing $t \geq 0$, the convergence of T_n to T yields $F \cap (T \leq t) = \cap_{n=1}^\infty \cup_{m=1}^\infty \cap_{k=m}^\infty F \cap (T_k \leq t + \frac{1}{n})$. Now, as $F \in \cap_{n=1}^\infty \mathcal{F}_{T_n}$, we have $F \in \mathcal{F}_{T_n}$ for all n. Therefore, $F \cap (T_k \leq t + \frac{1}{n}) \in \mathcal{F}_{t+\frac{1}{n}}$, and so $\cup_{m=1}^\infty \cap_{k=m}^\infty F \cap (T_k \leq t + \frac{1}{n}) \in \mathcal{F}_{t+\frac{1}{n}}$. As this sequence of sets is decreasing in n, we find that $\cap_{n=1}^\infty \cup_{m=1}^\infty \cap_{k=m}^\infty F \cap (T_k \leq t + \frac{1}{n})$ is in $\cap_{n=1}^\infty \mathcal{F}_{t+\frac{1}{n}}$. By right-continuity of the filtration, $\cap_{n=1}^\infty \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_t$, and so we finally conclude $F \cap (T \leq t) \in \mathcal{F}_t$, proving $F \in \mathcal{F}_T$. We have now shown $\cap_{n=1}^\infty \mathcal{F}_{T_n} \subseteq \mathcal{F}_T$, and so $\mathcal{F}_T = \cap_{n=1}^\infty \mathcal{F}_{T_n}$. This concludes the proof. □

Solution to exercise 1.5.8. Recall from Theorem 1.2.13 that M is a martingale. In order to show that M is not uniformly integrable, assume that M is in fact uniformly integrable, we seek a contradiction. We know by Theorem 1.2.4 that there is M_∞ such that M_t converges almost surely and in $\mathcal{L}^1$ to M_∞. However, we then obtain

$$\lim_{t \to \infty} E|\tfrac{1}{t} M_t| = \lim_{t \to \infty} \tfrac{1}{t} E|M_t| \leq \lim_{t \to \infty} \tfrac{1}{t} E|M_t - M_\infty| + \tfrac{1}{t} E|M_\infty| = 0,$$

so $\frac{1}{t} M_t$ converges in $\mathcal{L}^1$ to zero. In particular $\frac{1}{t} M_t$ converges in distribution to the Dirac measure at zero, which is in contradiction to the fact that as $\frac{1}{t} M_t = (\frac{1}{\sqrt{t}} W_t)^2 - 1$, $\frac{1}{t} M_t$ has the law of a standard normal distribution transformed by the transformation $x \mapsto x^2 - 1$ for any $t \geq 0$. Therefore, M_t cannot be convergent in $\mathcal{L}^1$, and then, again by Theorem 1.2.4, M cannot be uniformly integrable. □

Solution to exercise 1.5.9. By Lemma 1.2.8, it holds that if $M \in \mathcal{M}^u$, then M_T is integrable with $EM_T = 0$ for all stopping times T. It therefore suffices to prove the converse implication. Assume that M_T is integrable with $EM_T = 0$ for all stopping times T which take at most two values in $[0, \infty]$. Fix $t \geq 0$ and $F \in \mathcal{F}_t$. Then t_F is a stopping time taking the two values t and ∞, so $|M_{t_F}|$ is integrable and $EM_{t_F} = 0$. As we have $EM_{t_F} = E1_F M_t + E1_{F^c} M_\infty$ and we also have $EM_\infty = E1_F M_\infty + E1_{F^c} M_\infty$, we conclude $E1_F M_t = E1_F M_\infty$, leading to $M_t = E(M_\infty | \mathcal{F}_t)$. Thus, $M \in \mathcal{M}$ and so $M \in \mathcal{M}^u$ by Theorem 1.2.4. □

Solution to exercise 1.5.10. Fix α and let $0 \le s \le t$. We then find

$$\begin{aligned} E(M_t^\alpha|\mathcal{F}_s) &= M_s^\alpha E(\exp(\alpha(W_t - W_s) - \tfrac{1}{2}\alpha^2(t-s))|\mathcal{F}_s) \\ &= M_s^\alpha E(\exp(\alpha(W_t - W_s) - \tfrac{1}{2}\alpha^2(t-s))) = M_s^\alpha, \end{aligned}$$

using that $W_t - W_s$ is independent of $\mathcal{F}_s$ and normally distributed, and for any variable X which is standard normally distributed, $E\exp(tX) = \exp(\frac{1}{2}t^2)$ for any $t \in \mathbb{R}$. Thus, M^α is a martingale.

Next, we consider the result on the stopping time T. By symmetry, it suffices to consider the case where $a \ge 0$. By the law of the iterated logarithm, T is almost surely finite. Therefore, W^T is bounded from above by a and $W_T = a$. Fixing some $\alpha \ge 0$, we then find $(M^\alpha)_t^T = \exp(\alpha W_t^T - \frac{1}{2}\alpha^2(T \wedge t)) \le \exp(\alpha a)$. Therefore, by Lemma A.3.4 $(M^\alpha)^T$ is uniformly integrable. Now, the almost sure limit of $(M^\alpha)^T$ is $\exp(\alpha a - \frac{1}{2}\alpha^2 T)$. By Theorem 1.2.6, we therefore find $1 = E(M^\alpha)_\infty^T = E\exp(\alpha a - \frac{1}{2}\alpha^2 T) = \exp(\alpha a)E\exp(-\frac{1}{2}\alpha^2 T)$. This shows $E\exp(-\frac{1}{2}\alpha^2 T) = \exp(-\alpha a)$. Thus, if we now fix $\beta \ge 0$, we find

$$E\exp(-\beta T) = E\exp(-\tfrac{1}{2}(\sqrt{2\beta})^2 T) = \exp(-\sqrt{2\beta}a),$$

as desired. □

Solution to exercise 1.5.11. Fix $0 \le s \le t$. As $(W_t - W_s)^3 = W_t^3 - 3W_t^2 W_s + 3W_t W_s^2 - W_s^3$, we obtain

$$\begin{aligned} E(W_t^3 - 3tW_t|\mathcal{F}_s) &= E((W_t - W_s)^3|\mathcal{F}_s) + E(3W_t^2 W_s - 3W_t W_s^2 + W_s^3|\mathcal{F}_s) - 3tW_s \\ &= E(W_t - W_s)^3 + 3W_s E(W_t^2|\mathcal{F}_s) - 3W_s^2 E(W_t|\mathcal{F}_s) + W_s^3 - 3tW_s \\ &= 3W_s E(W_t^2 - t|\mathcal{F}_s) - 2W_s^3 = 3W_s(W_s^2 - s) - 2W_s^3 = W_s^3 - 3sW_s. \end{aligned}$$

As regards the second process, we have $(W_t - W_s)^4 = W_t^4 - 4W_t^3 W_s + 6W_t^2 W_s^2 - 4W_t W_s^3 + W_s^4$, and so

$$\begin{aligned} E(W_t^4|\mathcal{F}_s) &= E((W_t - W_s)^4|\mathcal{F}_s) + E(4W_t^3 W_s - 6W_t^2 W_s^2 + 4W_t W_s^3 - W_s^4|\mathcal{F}_s) \\ &= 3(t-s)^2 + 4W_s E(W_t^3|\mathcal{F}_s) - 6W_s^2 E(W_t^2|\mathcal{F}_s) + 4W_s^3 E(W_t|\mathcal{F}_s) - W_s^4 \\ &= 3(t-s)^2 + 4W_s E(W_t^3 - 3tW_t|\mathcal{F}_s) + 12tW_s^2 - 6W_s^2 E(W_t^2|\mathcal{F}_s) + 3W_s^4 \\ &= 3(t-s)^2 + 4W_s(W_s^3 - 3sW_s) + 12tW_s^2 - 6W_s^2 E(W_t^2|\mathcal{F}_s) + 3W_s^4 \\ &= 3(t-s)^2 + 12(t-s)W_s^2 - 6W_s^2 E(W_t^2 - t|\mathcal{F}_s) - 6tW_s^2 + 7W_s^4 \\ &= 3(t-s)^2 + 12(t-s)W_s^2 - 6W_s^2(W_s^2 - s) - 6tW_s^2 + 7W_s^4 \\ &= 3(t-s)^2 + 6(t-s)W_s^2 + W_s^4. \end{aligned}$$

Therefore, we find

$$\begin{aligned}
E(W_t^4 - 6tW_t^2 + 3t^2|\mathcal{F}_s) &= 3(t-s)^2 + 6(t-s)W_s^2 + W_s^4 - 6tE(W_t^2|\mathcal{F}_s) + 3t^2 \\
&= 3(t-s)^2 + 6(t-s)W_s^2 + W_s^4 - 6tE(W_t^2 - t|\mathcal{F}_s) - 3t^2 \\
&= 3(t-s)^2 + 6(t-s)W_s^2 + W_s^4 - 6t(W_s^2 - s) - 3t^2 \\
&= W_s^4 - 6sW_s^2 + 3(t-s)^2 + 6st - 3t^2 \\
&= W_s^4 - 6sW_s^2 + 3s^2,
\end{aligned}$$

as desired. □

Solution to exercise 1.5.12. We have $T = \inf\{t \geq 0 \mid W_t - a - bt \geq 0\}$, where the process $W_t - a - bt$ is continuous and adapted. Therefore, Exercise 1.5.3 shows that T is a stopping time. Now consider $a > 0$ and $b > 0$. Note that $W_t - a - bt$ has initial value $-a \neq 0$, so we always have $T > 0$. In particular, again using continuity, we have $W_T = a + bT$ whenever T is finite. Now, by Exercise 1.5.10, we know that for any $\alpha \in \mathbb{R}$, the process M^α defined by $M_t^\alpha = \exp(\alpha W_t - \frac{1}{2}\alpha^2 t)$ is a martingale. We then find

$$\begin{aligned}
E1_{(T<\infty)}M_T^\alpha &= E1_{(T<\infty)}\exp(\alpha W_T - \tfrac{1}{2}\alpha^2 T) \\
&= E1_{(T<\infty)}\exp(\alpha(a+bT) - \tfrac{1}{2}\alpha^2 T) \\
&= \exp(\alpha a)E1_{(T<\infty)}\exp(T(\alpha b - \tfrac{1}{2}\alpha^2)).
\end{aligned}$$

Now note that the equation $\alpha b = \frac{1}{2}\alpha^2$ has the unique nonzero solution $\alpha = 2b$. Therefore, we obtain $E1_{(T<\infty)}M_T^{2b} = \exp(2ab)P(T < \infty)$. In order to show the desired equality, it therefore suffices to prove $E1_{(T<\infty)}M_T^{2b} = 1$. To this end, note that by the law of the iterated logarithm, using that $b \neq 0$, $\lim_{t\to\infty} 2bW_t - \frac{1}{2}(2b)^2 t = \lim_{t\to\infty} t(2b\frac{W_t}{t} - 2b^2) = -\infty$, so that $\lim_{t\to\infty} M_t^{2b} = 0$. Therefore, $1_{(T<\infty)}M_T^{2b} = M_T^{2b}$ and it suffices to prove $EM_T^{2b} = 1$. To this end, note that

$$\begin{aligned}
M_{T\wedge t}^{2b} &= \exp\left(2bW_{T\wedge t} - \frac{1}{2}(2b)^2(T\wedge t)\right) \\
&\leq \exp\left(2b(a + b(T\wedge t)) - \frac{1}{2}(2b)^2(T\wedge t)\right) = \exp(2ba).
\end{aligned}$$

Therefore, $(M^{2b})^T$ is bounded by the constant $\exp(2ba)$, in particular it is uniformly integrable. Thus, Theorem 1.2.6 shows that $EM_T^{2b} = E(M^{2b})_\infty^T = 1$. From this, we finally conclude $P(T < \infty) = \exp(-2ab)$. □

Solution to exercise 1.5.13. Since $W_t^2 - a(1-t)$ is a continuous adapted process, Exercise 1.5.3 shows that T is a stopping time. Now let $a > 0$. As $W_t^2 - a(1-t)$ has initial value $-a \neq 0$, we always have $T > 0$. Furthermore, as $a(1-t) < 0$ when $t > 1$, we must have

$T \leq 1$, so T is a bounded stopping time. In particular, T has moments of all orders, and by continuity, $W_T^2 = a(1-T)$.

In order to find the mean of T, recall from Theorem 1.2.13 that $W_t^2 - t$ is a martingale. As T is a bounded stopping time, we find $0 = E(W_T^2 - T) = E(a(1-T) - T) = a - (1+a)ET$ by Theorem 1.2.6, so $ET = \frac{a}{1+a}$. Next, we consider the second moment. Recall by Exercise 1.5.11 that $W_t^4 - 6tW_t^2 + 3t^2$ is in $\mathbf{c}\mathcal{M}$. Again using Theorem 1.2.6, we obtain

$$\begin{aligned}
0 &= E(W_T^4 - 6TW_T^2 + 3T^2) \\
&= E(a^2(1-T)^2 - 6Ta(1-T) + 3T^2) \\
&= E(a^2(1-2T+T^2) + 6a(T^2-T) + 3T^2) \\
&= (a^2+6a+3)ET^2 - (2a^2+6a)ET + a^2.
\end{aligned}$$

Recalling that $ET = \frac{a}{1+a}$, we find

$$(2a^2+6a)ET - a^2 = \frac{a(2a^2+6a)}{1+a} - \frac{(1+a)a^2}{1+a} = \frac{a^3+5a^2}{1+a},$$

from which we conclude

$$ET^2 = \frac{a^3+5a^2}{(1+a)(a^2+6a+3)} = \frac{a^3+5a^2}{a^3+7a^2+9a+3},$$

concluding the solution to the exercise. □

Solution to exercise 1.5.14. As N is a Poisson process, it is immediate that M_t is integrable for all $t \geq 0$. Let $0 \leq s \leq t$. As $N_t - N_s$ is Poisson distributed with paramter $t-s$ and is independent of $\mathcal{F}_s$, we obtain, applying Theorem 1.2.15, that

$$\begin{aligned}
E(M_t|\mathcal{F}_s) &= E(N_t^2|\mathcal{F}_s) - 2tE(N_t|\mathcal{F}_s) + t^2 - t \\
&= E((N_t-N_s)^2|\mathcal{F}_s) + E(2N_tN_s - N_s^2|\mathcal{F}_s) - 2t(t+N_s-s) + t^2 - t \\
&= E(N_t-N_s)^2 + 2N_s(t+N_s-s) - N_s^2 - 2t(t+N_s-s) + t^2 - t \\
&= (t-s)^2 + N_s^2 + 2N_s(t-s) - 2t(t+N_s-s) + t^2 - s \\
&= N_s^2 + N_s(2(t-s)-2t) + (t-s)^2 + 2ts - t^2 - s \\
&= N_s^2 - 2sN_s + s^2 - s = M_s,
\end{aligned}$$

so M is a martingale, as desired. □

Solution to exercise 1.5.15. Fix $\alpha \in \mathbb{R}$. As N is a Poisson process, it is immediate that M_t^α

is integrable for all $t \geq 0$. Let $0 \leq s \leq t$. As $N_t - N_s$ is independent of $\mathcal{F}_s$, we obtain

$$\begin{aligned} E(M_t^\alpha|\mathcal{F}_s) &= E(\exp(\alpha N_t - (e^\alpha - 1)t)|\mathcal{F}_s) \\ &= E(\exp(\alpha(N_t - N_s))|\mathcal{F}_s)\exp(-(e^\alpha - 1)(t-s))\exp(\alpha N_s - (e^\alpha - 1)s) \\ &= E\exp(\alpha(N_t - N_s))\exp(-(e^\alpha - 1)(t-s))M_s^\alpha, \end{aligned}$$

and as $N_t - N_s$ is Poisson distributed with paramter $t - s$, we have

$$E\exp(\alpha(N_t - N_s)) = \exp((e^\alpha - 1)(t-s)),$$

which all in all yields $E(M_t^\alpha|\mathcal{F}_s) = M_s^\alpha$, as desired. □

C.2 Solutions for Chapter 2

Solution to exercise 2.4.1. By Lemma 1.1.4, every adapted càglàd process may be approximated by adapted càdlàg processes. Therefore, $\Sigma^p \subseteq \Sigma^o$. And by Lemma 1.1.6, every càdlàg adapted process is progressive, and therefore $\Sigma^o \subseteq \Sigma^\pi$. □

Solution to exercise 2.4.2. First note that by Lemma 1.5.3, T is a stopping time. We show that it is predictable. For each n, define $U_n = (a - 1/n, a + 1/n)$. Then U_n is an open set containing a. Define a mapping by putting $S_n = \inf\{t \geq 0 \mid X_t \in U_n\}$. By Lemma 1.1.13, S_n is a stopping time. Put $T_n = S_n \wedge n$, then (T_n) is a sequence of stopping times. we will argue that (T_n) is an announcing sequence for T.

To this end, first note that as the U_n are decreasing, the sequence (S_n) is increasing and thus the sequence (T_n) is increasing. Also, as $a \in U_n$, we have $T_n \leq S_n \leq T$ for all n. Note that as X has initial value zero and $a \neq 0$, we always have $T > 0$. Also, for n large enough, $[a - 1/n, a + 1/n]$ does not contain zero. For such n, $0 < S_n$, and therefore $X_{S_n} \in [a - 1/n, a + 1/n]$ for such n.

We first argue that S_n increases to T. Consider the case where T is finite, in particular S_n is finite for all n. Assume that $\lim_n S_n < T$. As $X_{S_n} \in [a - 1/n, a + 1/n]$ for n large enough, we obtain $X_{\lim_n S_n} = a$ by left-continuity, a contradiction with $\lim_n S_n < T$. Thus, $\lim_n S_n = T$. Next, consider the case where T is infinite. If $\lim_n S_n$ is finite, we again have $X_{S_n} \in [a - 1/n, a + 1/n]$ from a point onwards and so $X_{\lim_n S_n} = a$. Therefore, T is not infinite, a contradiction. We conclude that in all cases, S_n increases to T.

As S_n increases to T, so does T_n. Therefore, in order to show that (T_n) is an announcing sequence for T, it suffices to show that $T_n < T$. This is immediate when T is infinite, therefore, consider the case where T is finite, so that $X_T = a$. For n large enough, $0 < S_n$. As $X_t \notin (a - 1/n, a + 1/n)$ for $0 \le t < S_n$ and $X_{S_n} \in [a - 1/n, a + 1/n]$, we must either have $X_{S_n} = a - 1/n$ or $X_{S_n} = a + 1/n$ in this case. In particular, $S_n \neq T$, so $S_n < T$ and we thus obtain $T_n < T$.

We conclude that (T_n) is an announcing sequence for T, and so T is predictable. □

Solution to exercise 2.4.3. By Lemma 2.2.2, it is immediate that $\mathcal{F}_{(S \wedge T)-} \subseteq \mathcal{F}_{S-} \wedge \mathcal{F}_{T-}$. To prove the converse implication, assume that $F \in \mathcal{F}_{S-} \wedge \mathcal{F}_{T-}$. By Lemma 2.2.4, $F \cap (S \le T)$ and $F \cap (T \le S)$ are both $\mathcal{F}_{(S \wedge T)-}$ measurable. Therefore, F is $\mathcal{F}_{(S \wedge T)-}$ measurable as well, proving the result. □

Solution to exercise 2.4.4. It is immediate from Lemma 1.1.10 that T is a stopping time, so it suffices to show that T is predictable. Define $R_n = \max_{k \le n} T_k$. By Lemma 1.1.9, (R_n) is a sequence of stopping times. It is immediate that (R_n) is increasing and that $R_n < T$ whenever $T > 0$. Also,

$$\lim_n R_n = \lim_n \max_{k \le n} T_k = \sup_{n \ge 1} T_n = T,$$

so R_n is an announcing sequence for T, and thus T is predictable. □

Solution to exercise 2.4.5. Let (q_n) be an enumeration of $\mathbb{Q}_+$ and define

$$S_n = S_{(S + q_n \ge T)} \wedge (S + q_n)_{(S + q_n < T)}.$$

Note that as $(S + q_n \ge T) \in \mathcal{F}_T = \mathcal{F}_S$, the above is a stopping time by Lemma 1.1.9. Put $T_n = n \wedge S_n$. We claim that the sequence (T_n) satisfies the requirements of Exercise 2.4.4.

We first show that $T_n < T$ whenever $T > 0$. Thus, assume that $T > 0$. If T is infinite, it is immediate that $T_n < T$ as T_n is finite. If T is finite, then S is finite and thus $S < T$, so that $S_{(S + q_n \ge T)} < T$ if $S + q_n \ge T$ and $(S + q_n)_{(S + q_n < T)} < T$ if $S + q_n < T$. In both cases, we obtain $T_n \le S_n < T$.

Next, we show that $\sup_n T_n = T$. It is immediate that $T_n \le T$ for all n. In the case where T is infinite and S is infinite, we have $T_n = n$ and thus $\sup T_n = T$. In the case where T is infinite and S is finite, we obtain $S_n = S + q_n$ and $T_n = n \wedge (S + q_n)$. For any $k \ge 1$, we have $q_n \ge k$ infinitely often and $n \ge k$ eventually, yielding $T_n \ge k$ infinitely often and thus $\sup T_n = T$ in this case as well. Finally, consider the case where T is finite, so that S also is finite and $S < T$. Fix $\varepsilon > 0$ and $k \ge T$. There exists $n \ge k$ such that $T - \varepsilon < S + q_n < T$, and thus $T - \varepsilon \le T_n$. This shows $\sup_n T_n = T$ in this case as well.

The results of Exercise 2.4.4 now shows that T is a predictable stopping time. $\square$

Solution to exercise 2.4.6. By Lemma 2.2.6, it holds that X_T is $\mathcal{F}_{T-}$ measurable whenever X is predictable. From this, we obtain that $\sigma(\{X_T \mid X \text{ is predictable }\}) \subseteq \mathcal{F}_{T-}$. Conversely, fix some $F \in \mathcal{F}_{T-}$ and define $X = 1_F 1_{[\![T,\infty[\![}$. By Lemma 2.2.5, X is predictable, and $X_T = 1_F$. Therefore, $F \in \sigma(\{X_T \mid X \text{ is predictable }\})$. This allows us to conclude that $\mathcal{F}_{T-} \subseteq \sigma(\{X_T \mid X \text{ is predictable }\})$, proving the desired result. $\square$

Solution to exercise 2.4.7. By Lemma 2.2.7, X_{T-} is $\mathcal{F}_{T-}$ measurable whenever X is càdlàg adapted. From this, we obtain that $\sigma(\{X_{T-} \mid X \text{ is càdlàg adapted }\}) \subseteq \mathcal{F}_{T-}$. To prove the other inclusion, let $F \in \mathcal{F}_{T-}$ and define $X = 1_F 1_{[\![T,\infty[\![}$. Then X is càdlàg adapted, and $X_T = 1_F$. Therefore, $F \in \sigma(\{X_T \mid X \text{ is càdlàg adapted }\})$ and thus it holds that $\mathcal{F}_{T-} \subseteq \sigma(\{X_T \mid X \text{ is càdlàg adapted }\})$. This concludes the proof of the exercise. $\square$

Solution to exercise 2.4.8. Assume that T takes its values in the countable set $\{t_n \mid n \geq 1\}$. By Lemma 2.1.5, t_n is a predictable stopping time for each $n \geq 1$, and we have $[\![T]\!] \subseteq \cup_{n=1}^\infty [\![t_n]\!]$. Thus, T is accessible. $\square$

Solution to exercise 2.4.9. If T is predictable, Lemma 2.2.4 yields $(T = S) \in \mathcal{F}_{S-}$ for all predictable stopping times S. Assume conversely that T is accessible and that $(T = S) \in \mathcal{F}_{S-}$ for all predictable stopping times S. Let (S_n) be a sequence of predictable stopping times such that $[\![T]\!] \subseteq \cup_{n=1}^\infty [\![S_n]\!]$. Put $T_n = (S_n)_{(T=S_n)}$. By our assumptions, $(T = S_n) \in \mathcal{F}_{S_n-}$, so T_n is a predictable stopping time by Lemma 2.2.3, and T_n is either equal to T or infinity. Let $R_n = \min_{k\leq n} T_k$, by Lemma 2.1.5, (R_n) is a sequence of predictable stopping times. It is immediate that (R_n) is decreasing. If there is some k such that $T_k = T$, then R_n decreases to T and is constant from a point onwards. If there is no k such that $T_k = T$, then T is infinite, R_n is infinite for all n and thus is equal to T and also constant from a point onwards. By Lemma 2.1.9, T is a predictable stopping time. $\square$

Solution to exercise 2.4.10. As Lemma 1.2.8 provides that M_T is integrable with $EM_T = 0$ for all stopping times T whenever $M \in \mathcal{M}^u$, it suffices to prove the converse implication. However, as all stopping times taking only countably many values are accessible by Exercise 2.4.8, the desired result follows from Exercise 1.5.9. $\square$

Solution to exercise 2.4.11. Let $t_k^n = k2^{-n}$ for $n, k \geq 0$. Fix $n \geq 1$ and define a stopping time T_n by putting $T_n = \infty$ when $T = \infty$ and, for $k \geq 1$, $T_n = k2^{-n}$ when $t_{k-1}^n < T \leq t_k^n$. Then (T_n) is a sequence of stopping times taking values in the dyadic rationals $\mathbb{D}_+$ and infinity and converging downwards to T. By Exercise 2.4.8, each T_n is accessible. This proves the result. □

C.3 Solutions for Chapter 3

Solution to exercise 3.5.1. Since $X^{S\wedge T} = (X^S)^T$, we find by Lemma 1.2.7 that $X^{S\wedge T}$ is in $\mathcal{M}^u$. As $X_{S\wedge t} + X_{T\wedge t} = X_{(S\wedge T)\wedge t} + X_{(S\vee T)\wedge t}$, we find $X^{S\vee T} = X^S + X^T - X^{S\wedge T}$, so $X^{S\vee T}$ is in $\mathcal{M}^u$ as well, since it is a linear combination of elements in $\mathcal{M}^u$. □

Solution to exercise 3.5.2. If $M \in \mathcal{M}^u$, we have by Theorem 1.2.6 that $M_T = E(M_\infty|\mathcal{F}_T)$ for all $T \in \mathcal{C}$. Therefore, Lemma A.3.6 shows that $(M_T)_{T\in\mathcal{C}}$ is uniformly integrable. Conversely, assume that $(M_T)_{T\in\mathcal{C}}$ is uniformly integrable, we will use Lemma 1.2.8 to show that $M \in \mathcal{M}$. Let S be any bounded stopping time, and let (T_n) be a localising sequence with the property that $M^{T_n} \in \mathcal{M}^u$. As S is bounded, $M_{T_n\wedge S}$ converges almost surely to M_S. As it holds that $(M_{T_n\wedge S}) \subseteq (M_T)_{T\in\mathcal{C}}$, $(M_{T_n\wedge S})$ is uniformly integrable, and so Lemma A.3.5 shows that $M_{T_n\wedge S}$ converges in $\mathcal{L}^1$ to M_S. In particular, $EM_S = \lim_n EM_{T_n\wedge S} = \lim_n EM_S^{T_n} = 0$ by Theorem 1.2.6. Lemma 1.2.8 then shows that $M \in \mathcal{M}$. As $(M_t)_{t\geq 0} \subseteq (M_T)_{T\in\mathcal{C}}$, M is uniformly integrable, and so $M \in \mathcal{M}^u$ as well. □

Solution to exercise 3.5.3. Let (T_n) be a localising sequence for M. For $t \geq 0$, Fatou's lemma yields

$$EM_t = E\liminf_n M_t^{T_n} \leq \liminf_n EM_t^{T_n} = EM_0,$$

so M_t is integrable for all $t \geq 0$. Letting $0 \leq s \leq t$, we obtain

$$E(M_t|\mathcal{F}_s) = E(\liminf_n M_t^{T_n}|\mathcal{F}_s) \leq \liminf_n E(M_t^{T_n}|\mathcal{F}_s) = \liminf_n M_s^{T_n} = M_s,$$

so M is a supermartingale. □

Solution to exercise 3.5.4. First consider the case where $M \in \mathcal{M}^u$. Let $T_n = \inf\{t \geq 0 \mid |M_t| > n\}$, (T_n) is then a localising sequence. We then obtain

$$EM^*_{T_n} \leq n + E1_{(T_n<\infty)}|\Delta M_{T_n}| \leq n + E1_{(T_n<\infty)}|M_{T_n}| \leq n + E|M_{T_n}|,$$

which is finite. Thus, $(M^*)^{T_n} \in \mathcal{A}^i$ and so $M^* \in \mathcal{A}^i_\ell$ in this case. Consider the case of a general element $M \in \mathcal{M}_\ell$. Let (T_n) be a localising sequence. From what we already have shown, there exists for each $n \geq 1$ a localising sequence $(T_{nk})_{k\geq 1}$ such that $(M^*)^{T_{nk}} \in \mathcal{A}^i$. Let $S_n = \max_{k\leq n}\max_{i\leq n} T_{ki}$, then S_n is a localising sequence and $(M^*)_{S_n} \in \mathcal{A}^i$, so $M^* \in \mathcal{A}^i_\ell$, as was to be shown. □

Solution to exercise 3.5.5. First consider the case where $M \in \mathcal{M}^u$. Let T be a predictable stopping time. By Lemma 3.1.8, it holds that ΔM_T is integrable and $E(\Delta M_T | \mathcal{F}_{T-}) = 0$. As $\Delta M_t \geq 0$, this implies that ΔM_T is almost surely zero.

Now consider the case of a general $M \in \mathcal{M}_\ell$. Fix a predictable stopping time T. Let (T_n) be a localising sequence such that $M^{T_n} \in \mathcal{M}^u$ for $n \geq 1$. By what we already have shown, $\Delta M^{T_n}_T$ is almost surely zero. However, $\Delta M^{T_n}_T = 1_{(T \leq T_n)} \Delta M_T$. Letting n tend to infinity, this shows that $1_{(T<\infty)} \Delta M_T$ and thus ΔM_T is almost surely zero, as desired. □

Solution to exercise 3.5.6. Define $M_t = N_t - t$. By Theorem 1.2.15, $M \in \mathcal{M}_\ell$. Also, $\Delta M = \Delta N \geq 0$. Therefore, by Exercise 3.5.5, $\Delta M_T = 0$ almost surely for all predictable stopping times T. As $\Delta M_{T_n} = 1$, we therefore obtain for any predictable stopping time T that $P(T = T_n < \infty) \leq P(\Delta M_T = \Delta M_{T_n}) = 0$, so T_n is totally inaccessible. □

Solution to exercise 3.5.7. Let T_n be the n'th jump time of N, (T_n) is then a localising sequence and N^{T_n} is bounded, therefore $N^{T_n} \in \mathcal{A}^i$ and so $N \in \mathcal{A}^i_\ell$. As $N_t - t$ is in $\mathcal{M}_\ell$ by Theorem 1.2.15 and the process $(t, \omega) \mapsto t$ is predictable as it is continuous, it follows that $\Pi^*_p N_t = t$. □

Solution to exercise 3.5.8. First consider the case where $A \in \mathcal{A}^i$. By Lemma 3.2.4, $\Pi^*_p A$ is then in $\mathcal{A}^i$ as well and $A - \Pi^*_p A$ is in $\mathcal{M}^u$. Letting T be any predictable stopping time, Lemma 3.1.8 then shows that ΔM_T is integrable with $E(\Delta M_T | \mathcal{F}_{T-}) = 0$. As $\Pi^*_p A$ is almost surely continuous, $\Delta M_t = \Delta A_T$ almost surely. Thus, $E(\Delta A_T | \mathcal{F}_{T-}) = 0$ almost surely, which implies that ΔA_T is almost surely zero.

Now consider the case $A \in \mathcal{A}^i_\ell$. Letting (T_n) be a localising sequence, we then find that for any predictable stopping time T, $\Delta A^{T_n}_T$ is almost surely zero. Letting n tend to infinity, this implies that ΔA_T is almost surely zero. □

Solution to exercise 3.5.9. First consider the case where $A \in \mathcal{A}$. Define for $n \geq 1$ a mapping T_n by putting $T_n = \inf\{t \geq 0 \mid A_t \geq n\}$, then T_n is a predictable stopping time by Lemma 3.2.6, and it is positive as A has initial value zero. Let $(T_{nk})_{k\geq 1}$ be a localising sequence, it then always holds that $T_{nk} < T_n$ and thus $A^{T_{nk}}$ is bounded by n. Therefore, $A^{T_{nk}} \in \mathcal{A}^i$. As $\sup_n \sup_k T_{nk} = \sup_n T_n = \infty$, this shows that $A \in \mathcal{A}^i_\ell$.

Next, consider the case where $A \in \mathcal{V}$. We then have $A = A^+ - A^-$, where $A^+ = \frac{1}{2}(V_A + A)$ and $A^- = \frac{1}{2}(V_A - A)$. By Lemma 1.4.1, A^+ and A^- are in $\mathcal{A}$. And as V_A is càdlàg with $\Delta V_A = |\Delta A|$, Theorem 2.3.9 shows that V_A is predictable. Therefore, A^+ and A^- are both predictable, and therefore in $\mathcal{A}^i_\ell$. As a consequence, A is in $\mathcal{V}^i_\ell$, as desired. □

Solution to exercise 3.5.10. We first note that with T_n denoting the n'th jump time of N, both $N_-^{T_n}$ and N^{T_n} are bounded, so both $\int_0^t N_{s-}\,\mathrm{d}M_s$ and $\int_0^t N_s\,\mathrm{d}M_s$ are well-defined. Furthermore, as N_- is adapted and left-continuous, it is predictable. Therefore, Lemma 3.3.2 shows that $\int_0^t N_{s-}\,\mathrm{d}M_s$ is in $\mathcal{M}_\ell$. It remains to argue that the process $\int_0^t N_s\,\mathrm{d}M_s$ is not in $\mathcal{M}_\ell$. To this end, assume that $\int_0^t N_s\,\mathrm{d}M_s$ in fact is in $\mathcal{M}_\ell$, we seek a contradiction. First note that for all stopping times S, it holds that

$$\left|\int_0^{S\wedge T_n} N_s\,\mathrm{d}M_s\right| \le \int_0^{S\wedge T_n} |N_s|\,\mathrm{d}(V_M)_s \le N_{T_n}(N_{T_n}+T_n) = n(n+T_n),$$

and similarly, $|\int_0^{S\wedge T_n} N_{s-}\,\mathrm{d}M_s| \le n(n+T_n)$. As T_n is integrable, Exercise 3.5.2 shows that $\int_0^{t\wedge T_n} N_s\,\mathrm{d}N_s$ and $\int_0^{t\wedge T_n} N_{s-}\,\mathrm{d}N_s$ are in $\mathcal{M}^u$. However, we have

$$\begin{aligned}\int_0^{T_n} N_s\,\mathrm{d}M_s &= \int_0^{T_n} N_{s-}\,\mathrm{d}M_s + \int_0^{T_n} \Delta N_s\,\mathrm{d}M_s\\ &= \int_0^{T_n} N_{s-}\,\mathrm{d}M_s + \sum_{0<s\le T_n} \Delta N_s \Delta M_s\\ &= \int_0^{T_n} N_{s-}\,\mathrm{d}M_s + n,\end{aligned}$$

which by the uniformly integrable martingale property implies $E\int_0^{T_n} N_s\,\mathrm{d}M_s = n$, a contradiction. We conclude that the process $\int_0^t N_s\,\mathrm{d}M_s$ is not in $\mathcal{M}_\ell$. □

Solution to exercise 3.5.11. Fix $\lambda > 0$. As A has a single jump of size one at T, we have

$$E\exp(-\lambda \Pi_p^* A_T) = E\int_0^\infty \exp(-\lambda\Pi_p^* A_s)\,\mathrm{d}A_s.$$

Next, note that by Lemma 3.2.10, $\Pi_p^* A$ is almost surely continuous and has initial value zero. Therefore, $\exp(-\lambda\Pi_p^* A_s)$ is almost surely integrable with respect to any finite variation process and so, by Lemma 3.3.2, $\int_0^t \exp(-\lambda\Pi_p^* A_s)\,\mathrm{d}(A-\Pi_p^* A)_s$ is in $\mathcal{M}_\ell$. As $\Pi_p^* A \in \mathcal{A}^i$, we have $\exp(-\lambda\Pi_p^* A_s) \le 1$, and so the proof of Lemma 3.3.2 demonstrates that the integral process is in $\mathcal{M}^u$, yielding

$$E\int_0^\infty \exp(-\lambda\Pi_p^* A_s)\,\mathrm{d}A_s = E\int_0^\infty \exp(-\lambda\Pi_p^* A_s)\,\mathrm{d}\Pi_p^* A_s.$$

Next, note that as $A^T = A$, it holds that $A - \Pi_p^* A^T$ is in $\mathcal{M}^u$, and from this we conclude that $E\Pi_p^* A_T = EA_\infty = E\Pi_p^* A_\infty$. As $\Pi_p^* A_T \le \Pi_p^* A_\infty$, this implies $\Pi_p^* A_\infty = \Pi_p^* A_T$ almost surely.

Putting $\beta_s = \inf\{t \geq 0 \mid \Pi_p^* A_t \geq s\}$, we then obtain $(\beta_s < \infty) = (s \leq \Pi_p^* A_T)$. Applying Lemma A.2.14 and continuity of $\Pi_p^* A$, we then obtain

$$\begin{aligned}
\int_0^\infty \exp(-\lambda \Pi_p^* A_s)\, \mathrm{d}\Pi_p^* A_s &= \int_0^\infty \exp(-\lambda \Pi_p^* A_{\beta_s}) 1_{(\beta_s < \infty)}\, \mathrm{d}s \\
&= \int_0^{\Pi_p^* A_T} \exp(-\lambda s)\, \mathrm{d}s = \left[-\frac{1}{\lambda} \exp(-\lambda s) \right]_{s=0}^{s=\Pi_p^* A_T} \\
&= \frac{1}{\lambda} - \frac{1}{\lambda} E \exp(-\lambda \Pi_p^* A_T).
\end{aligned}$$

Collecting our conclusions and rearranging, this yields

$$E \exp(-\lambda \Pi_p^* A_T) = \frac{1}{\lambda} \left(1 + \frac{1}{\lambda} \right)^{-1} = \frac{1}{1+\lambda},$$

which is the Laplace transform of the exponential distribution. □

Solution to exercise 3.5.12. By Lemma 3.3.8, we have

$$[M^T - M^S] = [M^T] - 2[M^T, M^S] + [M^S] = [M]^T - 2[M]^S + [M^S] = [M]^T - [M]^S.$$

Therefore, if $[M]_S = [M]_T$ almost surely, we obtain that $[M^T - M^S]_\infty$ is almost surely zero. As $[M^T - M^S]$ is increasing, this implies that $[M^T - M^S]$ is evanescent, and so by Lemma 3.3.8, $M^T = M^S$ almost surely. □

Solution to exercise 3.5.13. Put $U_n = \sum_{k=1}^{2^n} (W_{t_k^n} - W_{t_{k-1}^n})^2$, we need to show that $U_n \xrightarrow{P} t$. By the properties of the Brownian motion W, the variables $W_{t_k^n} - W_{t_{k-1}^n}$ are independent and normally distributed with mean zero and variance $t2^{-n}$. Defining the variables Z_k^n by putting $Z_k^n = (\frac{1}{\sqrt{t2^{-n}}}(W_{t_k^n} - W_{t_{k-1}^n}))^2$, we find that Z_k^n, $k = 1, \ldots, 2^n$ are independent and distributed as an χ^2 distribution with one degree of freedom, and we have $U_n = t2^{-n} \sum_{k=1}^{2^n} Z_k^n$. As $EZ_k^n = 1$ and $VZ_k^n = 2$, we then obtain $EU_n = t$ and $VU_n = (t2^{-n})^2 \sum_{k=1}^{2^n} 2 = 2t^2 2^{-n}$. In particular, Chebychev's inequality yields for any $\varepsilon > 0$ that

$$\lim_{n\to\infty} P(|U_n - t| > \varepsilon) \leq \tfrac{1}{\varepsilon^2} \lim_{n\to\infty} E(U_n - t)^2 = \tfrac{1}{\varepsilon^2} \lim_{n\to\infty} VU_n = \tfrac{1}{\varepsilon^2} 2t^2 \lim_{n\to\infty} 2^{-n} = 0,$$

which proves that $U_n \xrightarrow{P} t$, as desired. The two convergences in probability then follow from the equalities

$$\begin{aligned}
W_t^2 &= 2\sum_{k=1}^{2^n} W_{t_{k-1}^n}(W_{t_k^n} - W_{t_{k-1}^n}) + \sum_{k=1}^{2^n} (W_{t_k^n} - W_{t_{k-1}^n})^2 \\
&= 2\sum_{k=1}^{2^n} W_{t_k^n}(W_{t_k^n} - W_{t_{k-1}^n}) - \sum_{k=1}^{2^n} (W_{t_k^n} - W_{t_{k-1}^n})^2,
\end{aligned}$$

by rearrangement and letting n tend to infinity. □

Solution to exercise 3.5.14. First assume that $M \in \mathcal{M}^2_\ell$. Let T_n be a localising sequence such that $M^{T_n} \in \mathcal{M}^2$. By Theorem 3.3.10, $[M^{T_n}] \in \mathcal{A}^i$ and therefore, by Lemma 3.3.8, $[M]^{T_n} \in \mathcal{A}^i$. Therefore, $[M] \in \mathcal{A}^i_\ell$.

Conversely, assume that $[M] \in \mathcal{A}^i_\ell$ and let (T_n) be a localising sequence such that $[M]^{T_n} \in \mathcal{A}^i$. As $[M^{T_n}] = [M]^{T_n}$ by Lemma 3.3.8, we obtain $[M^{T_n}] \in \mathcal{A}^i$ and thus $M^{T_n} \in \mathcal{M}^2$ by Theorem 3.3.10. This yields $M \in \mathcal{M}^2_\ell$, as desired. □

Solution to exercise 3.5.15. Let (T_n) be a localising sequence such that $M^{T_n} \in \mathcal{M}^2$ and $N^{T_n} \in \mathcal{M}^2$. By Theorem 3.3.10, $[M^{T_n}, N^{T_n}] \in \mathcal{V}^i$. By Lemma 3.3.8, this implies that $[M,N]^{T_n} \in \mathcal{V}^i$, so that $[M,N] \in \mathcal{V}^i_\ell$. □

Solution to exercise 3.5.16. If M is evanescent, Lemma 3.3.8 shows that $[M]$ is evanescent, so $\langle M \rangle$ is evanescent as well. Conversely, assume that $\langle M \rangle$ is evanescent. This implies that $[M] \in \mathcal{M}_\ell$. Let (T_n) be a localising sequence such that $[M]^{T_n}$ is in $\mathcal{M}^u$, we then obtain $E[M]_{T_n} = 0$ and thus $[M]_{T_n}$ is almost surely zero. Letting n tend to infinity, this implies that $[M]$ is evanescent and thus M is evanescent by Lemma 3.3.8. □

Solution to exercise 3.5.17. By Lemma 3.4.4, we have

$$[M]_t = \sum_{0<s\leq t} (\Delta M_s)^2 = \sum_{0<s\leq t} (\Delta N_s)^2 = \sum_{0<s\leq t} \Delta N_s = N_t,$$

as desired. □

Solution to exercise 3.5.18. By Lemma 3.4.6, there exists $M^d \in \mathbf{d}\mathcal{M}^2$ with $\Delta M^d = \Delta M$ almost surely. Letting M^c be a continuous modification of $M - M^d$, it is immediate that $M^c \in \mathbf{c}\mathcal{M}^2$ and $M = M^c + M^d$ almost surely. □

C.4 Solutions for Chapter 4

Solution to exercise 4.4.1. Assume given two decompositions $X = X_0 + M + A$ and $X = X_0 + N + B$, where $M, N \in \mathcal{M}_\ell$ and $A, B \in \mathcal{V}$ with A and B predictable. Then $M - N = B - A$, where $M - N \in \mathcal{M}_\ell$ and $B - A \in \mathcal{V}$ and $B - A$ is predictable. Thus, $M - N$ is a predictable element of $\mathcal{M}_\ell$, thus evanescent by Theorem 3.1.9. It follows that M and N are indistinguishable and A and B are indistinguishable, as desired. □

Solution to exercise 4.4.2. First assume that $X \in \mathcal{S}_p$, such that $X = M + A$ where $M \in \mathcal{M}_\ell$ and $A \in \mathcal{V}$ is predictable. By Exercise 3.5.4, $M^* \in \mathcal{A}^i_\ell$, and by Lemma 4.2.3, $A^* \in \mathcal{A}^i_\ell$. As $X^* \le M^* + A^*$, it follows that $X^* \in \mathcal{A}^i_\ell$ as well.

Conversely, assume that $X^* \in \mathcal{A}^i_\ell$. Let $X = M + A$ where $M \in \mathcal{M}_\ell$ and $A \in \mathcal{V}$. As $M^* \in \mathcal{A}^i_\ell$ by Exercise 3.5.4, we conclude that $A^* \in \mathcal{A}^i_\ell$ as well. Therefore, $A \in \mathcal{V}^i_\ell$, and so the compensator of A is well-defined. We may then write $X = M + (A - \Pi^*_p A) + \Pi^*_p A$, where $M + (A - \Pi^*_p A) \in \mathcal{M}_\ell$ and $\Pi^*_p A$ is a predictable element of $\mathcal{V}$, proving that $X \in \mathcal{S}_p$. □

Solution to exercise 4.4.3. If $X \in \mathcal{S}_p$, we have $X = X_0 + M + A$ where $M \in \mathcal{M}_\ell$ and A is predictable in $\mathcal{V}$. Then $X^T = X_0 + M^T + A^T$ for any stopping time T, where $M^T \in \mathcal{M}_\ell$ by Lemma 3.1.3 and A is predictable by Lemma 2.2.8. Therefore, taking for example $T_n = n$, we obtain that (T_n) is a localising sequence such that $X^{T_n} \in \mathcal{S}_p$ for all $n \ge 1$.

Conversely, assume that (T_n) is a localising sequence with $X^{T_n} \in \mathcal{S}_p$ for all $n \ge 1$. We then obtain $X^{T_n} = X_0 + M^n + A^n$, where $M^n \in \mathcal{M}_\ell$ and $A^n \in \mathcal{V}$ with A^n predictable. By Exercise 4.4.1, we obtain that $(A^{n+1})^{T_n} = A^n$ almost surely. Therefore, we may paste the processes A^n together to a process A which is a predictable element of $\mathcal{V}$ such that for all $n \ge 1$, $A^{T_n} = A^n$ almost surely. Likewise, the processes M^n may be pasted together to a process $M \in \mathcal{M}_\ell$ satisfying $M^{T_n} = M^n$. We now have $X = X_0 + M + A$ almost surely, where $M \in \mathcal{M}_\ell$ and $A \in \mathcal{V}$ with A predictable. Letting $N = X - X_0 - M - A$, we obtain that $N \in \mathcal{M}_\ell$, so $X = X_0 + (M + N) + A$ and we conclude $X \in \mathcal{S}_p$. □

Solution to exercise 4.4.4. If $X = X_0 + M + A$ with $M \in \mathbf{c}\mathcal{M}_\ell$ and $A \in \mathcal{V}$, it is immediate that X is predictable. Assume conversely that X is predictable. Put $Y = X - X_0$, Y is then predictable as well. By Lemma 4.2.3, $Y^* \in \mathcal{A}^i_\ell$, and so Exercise 4.4.2 shows that $Y \in \mathcal{S}_p$. This yields $Y = M + A$ where A is predictable, and so M is predictable. By Theorem 3.1.9, M is then almost surely continuous. Thus, we obtain $X = X_0 + M + A$ where $M \in \mathcal{M}_\ell$ is almost surely continuous and $A \in \mathcal{V}$ with A predictable. □

Solution to exercise 4.4.5. First assume that $X = X_0 + M + A$ with $M \in \mathcal{M}_\ell$ almost surely continuous. Then $\Delta X = \Delta A$ almost surely. As $A \in \mathcal{V}$, it almost surely holds for all $t \ge 0$ that $\sum_{0<s\le t} |\Delta A_s|$ and thus $\sum_{0<s\le t} |\Delta X_s|$ is almost surely convergent.

Conversely, assume that for all $t \ge 0$, $\sum_{0<s\le t} |\Delta X_s|$ is almost surely convergent. As $A \in \mathcal{V}$, we also have that for all $t \ge 0$, $\sum_{0<s\le t} |\Delta X_s|$ is almost surely convergent. Therefore, for all $t \ge 0$, $\sum_{0<s\le t} |\Delta M_s|$ is almost surely convergent, yielding $\Delta M \in \mathcal{V}$. Recall by Theorem 3.4.7 that $M = M^c + M^d$ almosts surely for some $M^c \in \mathbf{c}\mathcal{M}_\ell$ and $M^d \in \mathbf{d}\mathcal{M}_\ell$. As $\Delta M^d = \Delta M^c$,

Theorem 3.4.11 shows that $M^d \in \mathbf{fv}\mathcal{M}_\ell \subseteq \mathcal{V}$. We may thus decompose

$$X = X_0 + M^c + (M^d + A)$$

almost surely, where $M^d + A \in \mathcal{V}$. Modifying M^c on a null set, we obtain the result. □

Solution to exercise 4.4.6. It is immediate that if $X = X_0 + M + A$ with M and A continuous, then X is continuous as well. Conversely, assume that X is continuous. We then obtain $\Delta M = -\Delta A$, so Theorem 3.4.11 yields $M^d \in \mathbf{fv}\mathcal{M}_\ell$. We then obtain $X = X_0 + M^c + (M^d + A)$ almost surely, where $M^d + A \in \mathcal{V}$. Modifying M^c and $M^d + A$ on a null set, we obtain the result. □

Solution to exercise 4.4.7. First consider the case where $M \in \mathcal{M}^u$. We always have the relationship $M^{T-} = M^T - \Delta M_T 1_{[\![T,\infty[\![}$. Here, $M^T \in \mathcal{M}_\ell$ by Lemma 3.1.3 and $\Delta M_T 1_{[\![T,\infty[\![} \in \mathcal{M}_\ell$ by Lemma 3.1.8 and Lemma 3.2.10. This proves the result in the case where $M \in \mathcal{M}^u$. Now consider the case $M \in \mathcal{M}_\ell$. Let (T_n) be a localising sequence such that $M^{T_n} \in \mathcal{M}^u$. We then obtain

$$\begin{aligned}
(M^{T-})^{T_n}_t &= M^T_{t\wedge T_n} - \Delta M_T 1_{(t\wedge T_n \geq T)} \\
&= (M^{T_n})^T_t - \Delta M_T 1_{(t\geq T)} 1_{(T_n \geq T)} \\
&= (M^{T_n})^T_t - \Delta M^{T_n}_T 1_{(t\geq T)} = (M^{T_n})^{T-},
\end{aligned}$$

and so, by what we already have shown, $(M^{T-})^{T_n} \in \mathcal{M}_\ell$, yielding $M^{T-} \in \mathcal{M}_\ell$. □

Solution to exercise 4.4.8. By Theorem 4.2.8 and Lemma 4.2.11, we find that $H \cdot M \in \mathcal{M}_\ell$ and $[H \cdot M] = H^2 \cdot [M]$. Therefore, Theorem 3.3.10 yields the result. □

Solution to exercise 4.4.9. By Exercise 4.4.8, we have that $(H \cdot M)^t$ is in $\mathcal{M}^2$ for all $t \geq 0$ and that $E(H \cdot M)^2_t = E\int_0^t H^2_s \, \mathrm{d}[M]_s$ for all $t \geq 0$. In particular, this shows that $H \cdot M \in \mathcal{M}$. □

Solution to exercise 4.4.10. As $[W]_t = t$ by Theorem 3.3.6, this is immediate from Exercise 4.4.8 and Exercise 4.4.9. □

Solution to exercise 4.4.11. As $A - \Pi^*_p A$ is in $\mathcal{M}_\ell$ by Theorem 3.2.3, $H \cdot (A - \Pi^*_p A)$ is in $\mathcal{M}_\ell$ by Theorem 4.2.8, so the result follows by Theorem 3.2.3. □

Solution to exercise 4.4.12. Put $M_t = N_t - t$, then $M \in \mathcal{M}_\ell$ by Theorem 1.2.15, and so by Theorem 4.2.8, $H \cdot M \in \mathcal{M}_\ell$. Using Lemma 4.2.11 and Theorem 4.2.9, we have

$$(H \cdot M)_t = \int_0^t H_s \, \mathrm{d}N_s - \int_0^t H_s \, \mathrm{d}s = \sum_{k=1}^{N_t} H_{T_k} - \int_0^t H_s \, \mathrm{d}s,$$

and thus,

$$\frac{1}{t}\int_0^t H_s \,\mathrm{d}s = \frac{1}{t}(H\cdot M)_t + \frac{N_t}{t}\frac{1}{N_t}\sum_{k=1}^{N_t} H_{T_k}.$$

Now let $c > 0$ be a bound for H and note that $[H\cdot M]_t = \int_0^t H_s^2 \,\mathrm{d}[M]_s \leq c^2 N_t$. Therefore, $E[H\cdot M]_t$ is finite for all $t > 0$ and so, by Exercise 4.4.9, $E(H\cdot M)_t^2 = E[H\cdot M]_t \leq c^2 N_t$, and so

$$E\left(\frac{1}{t}(H\cdot M)_t\right)^2 = \frac{1}{t^2}E(H\cdot M)_t^2 \leq \frac{c^2}{t^2}EN_t = \frac{c^2}{t},$$

proving that $\frac{1}{t}(H\cdot M)_t$ converges in probability to zero as t tends to infinity. As we also have

$$E\left(\frac{N_t}{t} - 1\right)^2 = \frac{1}{t^2}E(N_t - t)^2 = \frac{1}{t^2}E[M]_t = \frac{1}{t},$$

we conclude that $N_t/t \xrightarrow{P} 1$. Collecting our conclusions, the result follows. □

Solution to exercise 4.4.13. As A has zero continuous martingale part, it is immediate that almost surely for all $t \geq 0$, $[M,A]_t = \sum_{0<s\leq t}\Delta M_s \Delta A_s$.

It remains to show that $[M,A] \in \mathcal{M}_\ell$. By Theorem 3.3.1, it suffices to prove this in the cases $M \in \mathcal{M}_\ell^b$ and $M \in \mathbf{fv}\mathcal{M}_\ell$. If $M \in \mathbf{fv}\mathcal{M}_\ell$, we have $[M,A]_t = \int_0^t \Delta A_s \,\mathrm{d}M_s$. Note that ΔA is predictable, and by arguments similar to those employed in the solution of Exercise 3.5.9, locally bounded. Therefore, Lemma 3.3.2 yields the result in this case. It remains to consider the case $M \in \mathcal{M}_\ell^b$.

To this end, we first consider the case where $M \in \mathcal{M}^b$ and $A \in \mathcal{A}^i$ with A predictable. In this case, ΔM is almost surely integrable on $[0,\infty)$ with respect to A, and as it holds that $\sum_{0<s\leq t}|\Delta M_s \Delta A_s| = \int_0^t |\Delta M_s| \,\mathrm{d}A_s$, this implies $[M,A] \in \mathcal{V}^i$. For each $t \geq 0$, define $T_t = \inf\{s \geq 0 | A_s \geq t\}$. By Lemma 3.2.6, T_t is a predictable stopping time. In particular, as $(T_t < \infty)$ is in $\mathcal{F}_{T_t}$, Lemma 3.1.8 shows that $E\Delta M_{T_t}1_{(T_t<\infty)} = 0$ and so, applying Lemma A.2.14, we obtain

$$E[M,A]_\infty = E\int_0^\infty \Delta M_t \,\mathrm{d}A_t = E\int_0^\infty \Delta M_{T_t}1_{(T_t<\infty)} \,\mathrm{d}t = \int_0^\infty E\Delta M_{T_t}1_{(T_t<\infty)} \,\mathrm{d}t = 0.$$

By Lemma 2.2.8, A^T is also predictable for any stopping time T, so the above yields $E[M,A]_T = E[M,A^T]_\infty = 0$ for all stopping times T, and so Lemma 1.2.8 shows that $[M,A]$ is in $\mathcal{M}^u$ in this case.

Next, consider the case where $M \in \mathcal{M}_\ell^b$ and $A \in \mathcal{A}$ with A predictable. By Lemma 4.2.3, there is a localising sequence (T_n) such that $M^{T_n} \in \mathcal{M}^b$ and $A^{T_n} \in \mathcal{A}^i$. We then obtain that $[M,A]^{T_n} = [M^{T_n}, A^{T_n}] \in \mathcal{M}_\ell$, so $[M,A] \in \mathcal{M}_\ell$ as well.

Finally, consider the case where $M \in \mathcal{M}_\ell^b$ and $A \in \mathcal{V}$ with A predictable. Then $A = A^+ - A^-$ with $A^+ = \frac{1}{2}(V_A + A)$ and $A^- = \frac{1}{2}(V_A - A)$ where $A^+, A^- \in \mathcal{A}$ and both processes are predictable. We thus have $[M, A^+], [M, A^-] \in \mathcal{M}_\ell$ and so $[M, A] = [M, A^+] - [M, A^-] \in \mathcal{M}_\ell$, as desired. □

Solution to exercise 4.4.14. Note that as $A \in \mathcal{V}^i$ and $M \in \mathcal{M}^b$, it holds that $M \cdot A$ is in $\mathcal{V}^i$, so the compensator of $\int_0^t M_s \, \mathrm{d}A_s$ is well-defined. Furthermore,

$$\int_0^t M_s \, \mathrm{d}A_s - \int_0^t M_{s-} \, \mathrm{d}A_s = \int_0^t \Delta M_s \, \mathrm{d}A_s = \sum_{0<s\le t} \Delta M_s \Delta A_s,$$

and the latter is in $\mathcal{M}_\ell$ by Exercise 4.4.13. Therefore, Theorem 3.2.3 yields the result. □

Solution to exercise 4.4.15. Note that as X and H are semimartingales, it holds that the sum $\sum_{0<s\le t} \Delta H_s \Delta X_s$ almost surely is absolutely convergent for all $t \ge 0$ by Lemma 4.1.11. Let $X = X_0 + M + A$ be a decomposition of X, where $M \in \mathcal{M}_\ell$ and $A \in \mathcal{V}$. Lemma 4.2.11 and Theorem 4.2.9 then yields

$$\begin{aligned} \int_0^t \Delta H_s \, \mathrm{d}X_s &= \int_0^t \Delta H_s \, \mathrm{d}M_s + \int_0^t \Delta H_s \, \mathrm{d}A_s \\ &= \int_0^t \Delta H_s \, \mathrm{d}M_s + \sum_{0<s\le t} \Delta H_s \Delta A_s. \end{aligned}$$

Also, by Exercise 4.4.4, $H = X_0 + N + B$ almost surely where $N \in \mathbf{c}\mathcal{M}_\ell$ and $B \in \mathcal{V}$ with B predictable. We then obtain $\int_0^t \Delta H_s \, \mathrm{d}M_s = \int_0^t \Delta B_s \, \mathrm{d}M_s$. Note that for any $N \in \mathbf{c}\mathcal{M}_\ell$, $[\Delta B \cdot M, N] = \Delta B \cdot [M, N] = 0$ as $[M, N]$ is continuous. Thus, $\Delta B \cdot M \in \mathbf{d}\mathcal{M}_\ell$. Next, put $L_t = \sum_{0<s\le t} \Delta B_s \Delta M_s$. By Exercise 4.4.13, $L \in \mathcal{M}_\ell$. Furthermore, as L has paths of finite variation, $L \in \mathbf{d}\mathcal{M}_\ell$ as well by Lemma 3.4.5. Finally, note that $\Delta(\Delta B \cdot M) = \Delta M \Delta M = \Delta L$. Combining our conclusions, Lemma 3.4.3 shows that $L - \Delta B \cdot M$ is evanescent so that $\Delta B \cdot M = L$. Recalling our earlier observations, we may now conclude

$$\int_0^t \Delta H_s \, \mathrm{d}X_s = \sum_{0<s\le t} \Delta B_s \Delta M_s + \sum_{0<s\le t} \Delta H_s \Delta A_s = \sum_{0<s\le t} \Delta H_s \Delta X_s,$$

as desired. □

Solution to exercise 4.4.16. Note that the conclusion is well-defined, as $W_{t+h} - W_t$ is almost surely never zero. To show the result, first note that Lemma 4.2.11 yields, up to indistinguishability,

$$
\begin{aligned}
\int_t^{t+h} H_s \, \mathrm{d}W_s &= \int_t^{t+h} H_s 1_{[\![t,\infty[\![} \, \mathrm{d}W_s = \int_t^{t+h} H_t 1_{[\![t,\infty[\![} \, \mathrm{d}W_s + \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![} \, \mathrm{d}W_s \\
&= H_t(W_{t+h} - W_t) + \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![} \, \mathrm{d}W_s,
\end{aligned}
$$

where the indicators $[\![t,\infty[\![$ are included as a formality to ensure that the integrals are well-defined. Therefore, it suffices to show that $(W_{t+h} - W_t)^{-1} \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![} \, \mathrm{d}W_s$ converges in probability to zero as h tends to zero. To this end, let c be a bound for H and note that by Exercise 4.4.10, we have

$$
E\left(\frac{1}{\sqrt{h}} \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![} \, \mathrm{d}W_s \right)^2 = \frac{1}{h} E \int_t^{t+h} (H_s - H_t)^2 \, \mathrm{d}s.
$$

As H is bounded and continuous, the dominated convergence theorem shows that the above tends to zero as h tends to zero. Thus, $\frac{1}{\sqrt{h}} \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![} \, \mathrm{d}W_s$ tends to zero in $\mathcal{L}^2$. Now fix $\delta, M > 0$. With $Y_h = \frac{1}{\sqrt{h}} \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![} \, \mathrm{d}W_s$, we have

$$
\begin{aligned}
& P\left(\left| (W_{t+h} - W_t)^{-1} \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![} \, \mathrm{d}W_s \right| > \delta \right) \\
\leq \; & P\left(\left| \frac{\sqrt{h}}{W_{t+h} - W_t} Y_h \right| > \delta, \left| \frac{\sqrt{h}}{W_{t+h} - W_t} \right| \leq M \right) + P\left(\left| \frac{\sqrt{h}}{W_{t+h} - W_t} \right| > M \right) \\
\leq \; & P\left(|Y_h| > \frac{\delta}{M} \right) + P\left(\left| \frac{\sqrt{h}}{W_{t+h} - W_t} \right| > M \right).
\end{aligned}
$$

Here, $P(\sqrt{h}(W_{t+h} - W_t)^{-1} > M)$ does not depend on h, as $(W_{t+h} - W_t)(\sqrt{h})^{-1}$ is a standard normal distribution. For definiteness, we define $\varphi(M) = P(\sqrt{h}(W_{t+h} - W_t)^{-1} > M)$. The above then allows us to conclude

$$
\limsup_{h \to \infty} P\left(\left| (W_{t+h} - W_t)^{-1} \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![} \, \mathrm{d}W_s \right| > \delta \right) \leq \varphi(M),
$$

and letting M tend to infinity, we obtain the desired result. □

Solution to exercise 4.4.17. For $q > p$, we have

$$
\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^q \leq \left(\max_{k \leq 2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^{q-p} \right) \sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^p.
$$

As X has continuous paths, the paths of X are uniformly continuous on $[0,t]$. In particular, $\max_{k \leq 2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^{q-p}$ converges almost surely to zero. Therefore, this variable also converges to zero in probability, and so $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^q$ converges in probability to zero, as was to be proven. $\square$

Solution to exercise 4.4.18. First, consider the case where $H = \frac{1}{2}$. In this case, X is a process whose finite-dimensional distributions are normally distributed with mean zero and with the property that for any $s, t \geq 0$, $EX_sX_t = \frac{1}{2}(t+s-|t-s|)$. For a Brownian motion W, we have that when $0 \leq s \leq t$, $EW_sW_t = EW_s(W_t - W_s) + EW_s^2 = EW_sE(W_t - W_s) + EW_s^2 = s$. In this case, $|t-s| = t-s$, and so $EW_sW_t = \frac{1}{2}(t+s-|t-s|)$. In the case where $0 \leq t \leq s$, $EW_sW_t = t = \frac{1}{2}(t+s+(t-s)) = \frac{1}{2}(t+s-|t-s|)$ as well. Thus, X and W are processes whose finite-dimensional distributions are normally distributed and have the same mean and covariance structure. Therefore, their distributions are the same, and so X has the distribution of a Brownian motion.

In order to show that X is not in $\mathbf{c}\mathcal{S}$ when $H \neq \frac{1}{2}$, we first fix $t \geq 0$ and consider the sum $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^p$ for $p \geq 0$. Understanding the convergence of such sums will allow us to prove our desired result. We know that the collection of variables $X_{t_k^n} - X_{t_{k-1}^n}$ follows a multivariate normal distribution with $E(X_{t_k^n} - X_{t_{k-1}^n}) = 0$ and, using the property that $EX_sX_t = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$, we obtain

$$\begin{aligned} E(X_{t_k^n} - X_{t_{k-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) &= EX_{t_k^n}X_{t_i^n} - EX_{t_k^n}X_{t_{i-1}^n} - EX_{t_{k-1}^n}X_{t_i^n} + EX_{t_{k-1}^n}X_{t_{i-1}^n} \\ &= 2^{-2nH}\tfrac{1}{2}(|k-i+1|^{2H} + |k-1-i|^{2H} - 2|k-i|^{2H}). \end{aligned}$$

Here, the parameter n only enters the expression through the constant multiplicative factor 2^{-2nH}. Therefore, as normal distributions are determined by their covariance structure, it follows that the distribution of the variables $(X_{t_k^n} - X_{t_{k-1}^n})$ for $k \leq 2^n$ is the same as the distribution of the variables $2^{-nH}(X_k - X_{k-1})$ for $k \leq 2^n$. In particular, it follows that the distributions of $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^p$ and $2^{-npH}\sum_{k=1}^{2^n} |X_k - X_{k-1}|^p$ are the same. We wish to apply the ergodic theorem for stationary processes to the sequence $(X_k - X_{k-1})_{k\geq 1}$. To this end, we first check that this sequence is in fact stationary. To do so, we need to check for any $m \geq 1$ that the variables $X_k - X_{k-1}$ for $k \leq n$ have the same distribution as the variables $X_{m+k} - X_{m+k-1}$ for $k \leq n$. As both families of variables are normally distributed with mean zero, it suffices to check that the covariance structure is the same. However, by what we already have shown,

$$\begin{aligned} & E(X_k - X_{k-1})(X_{t_i^n} - X_{i-1}) \\ = \;& \tfrac{1}{2}(|k-i+1|^{2H} + |k-1-i|^{2H} - 2|k-i|^{2H}) \\ = \;& E(X_{m+k} - X_{m+k-1})(X_{t_{m+i}^n} - X_{m+i-1}). \end{aligned}$$

This allows us to conclude that the sequence $(X_k - X_{k-1})_{k\geq 1}$ is stationary. As $E|X_k - X_{k-1}|^p$ is finite, the ergodic theorem shows that $\frac{1}{n}\sum_{k=1}^{n} |X_k - X_{k-1}|^p$ converges almost surely and in $\mathcal{L}^1$ to some variable Z_p, where Z_p is integrable and $EZ_p = E|X_1 - X_0|^p = E|X_1|^p > 0$. This property ensures that Z_p is not almost surely zero. Next, we observe that we have $2^{-npH}\sum_{k=1}^{2^n}|X_k - X_{k-1}|^p = 2^{n(1-pH)}(\frac{1}{2^n}\sum_{k=1}^{2^n}|X_k - X_{k-1}|^p)$, where we have just checked that the latter factor always converges almost surely and in $\mathcal{L}^1$ to Z_p. Having this result at hand, we are ready to prove that X is not in $\mathbf{c}\mathcal{S}$ when $H \neq \frac{1}{2}$.

First consider the case where $H < \frac{1}{2}$. In this case, $\frac{1}{H} > 2$. If $X \in \mathbf{c}\mathcal{S}$, we have that $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|^2$ converges in probability to $[X]_t$. Therefore, by Exercise, 4.4.17, we find that $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|^{\frac{1}{H}}$ converges to zero in probability. As $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|^{\frac{1}{H}}$ has the same distribution as $\frac{1}{2^n}\sum_{k=1}^{2^n}|X_k - X_{k-1}|^{\frac{1}{H}}$, we conclude that this sequence converges to zero in probability. However, this is in contradiction with what we have already shown, namely that this sequence converges in probability to a variable $Z_{\frac{1}{H}}$ which is not almost surely zero. We conclude that in the case $H < \frac{1}{2}$, X cannot be in $\mathbf{c}\mathcal{S}$.

Next, consider the case $H > \frac{1}{2}$. Again, we assume that $X \in \mathbf{c}\mathcal{S}$ and hope for a contradiction. In this case, $1 - 2H < 0$, so $2^{n(1-2H)}$ converges to zero and so, by our previous results, $2^{n(1-2H)}(\frac{1}{2^n}\sum_{k=1}^{2^n}|X_k - X_{k-1}|^2)$ converges to zero in probability. Therefore, we find that $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|^2$ converges to zero in probability as well, since this sequence has the same distribution as the previously considered sequence. By Theorem 4.3.3, this implies $[X]_t = 0$ almost surely. As $t \geq 0$ was arbitrary, we conclude that $[X]$ is evanescent. With $X = M + A$ being the decomposition of X into its continuous local martingale part and its continuous finite variation part, we have $[X] = [M]$, so $[M]$ is evanescent and so by Lemma 3.3.8, M is evanescent. Therefore, X almost surely has paths of finite variation. In particular, $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|$ is almost surely convergent, in particular convergent in probability. As $H < 1$, we have $\frac{1}{H} > 1$, so by Exercise 4.4.17, $\sum_{k=1}^{2^n}|X_{t_k^n} - X_{t_{k-1}^n}|^{\frac{1}{H}}$ converges in probability to zero. Therefore, $\frac{1}{2^n}\sum_{k=1}^{2^n}|X_k - X_{k-1}|^{\frac{1}{H}}$ converges to zero in probability as well. As in the previous case, this is in contradiction with the fact that that this sequence converges in probability to a variable $Z_{\frac{1}{H}}$ which is not almost surely zero. We conclude that in the case $H < \frac{1}{2}$, X cannot be in $\mathbf{c}\mathcal{S}$ either. □

Solution to exercise 4.4.19. By Itô's formula of Theorem 4.3.5 and Theorem 3.3.6, we have

$$\begin{aligned} f(W_t) - f(0) &= \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(W_s)\,\mathrm{d}W_s^i + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(W_s)\,\mathrm{d}[W^i, W^j]_s \\ &= \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(W_s)\,\mathrm{d}W_s^i + \frac{1}{2}\sum_{i=1}^p \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(W_s)\,\mathrm{d}s, \end{aligned}$$

and by our assumptions on f, this is equal to $\sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(W_s)\,\mathrm{d}W_s^i$, since the second term vanishes. Here, $\sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(W_s)\,\mathrm{d}W_s^i$ is in $\mathbf{c}\mathcal{M}_\ell$. Therefore, $f(W_t) - f(0)$ is in $\mathbf{c}\mathcal{M}_\ell$ and so $f(W_t)$ is a continuous local martingale. □

Solution to exercise 4.4.20. Define the two-dimensional process X by putting $X_t = (t, W_t)$. With $A_t = t$, we have $[A, W]_t = 0$, so Itô's formula of Theorem 4.3.5 shows

$$\begin{aligned} f(t,W_t) - f(0,0) &= \int_0^t \frac{\partial f}{\partial t}(s,W_s)\,\mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(s,W_s)\,\mathrm{d}W_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,W_s)\,\mathrm{d}s \\ &= \int_0^t \frac{\partial f}{\partial t}(s,W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,W_s)\,\mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(s,W_s)\,\mathrm{d}W_s, \end{aligned}$$

which is equal to $\int_0^t \frac{\partial f}{\partial x}(s,W_s)\,\mathrm{d}W_s$ by our assumptions on f, and this is in $\mathbf{c}\mathcal{M}_\ell$. Therefore, $f(t,W_t)$ is a continuous local martingale, and $f(t,W_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s,W_s)\,\mathrm{d}W_s$. □

Solution to exercise 4.4.21. As f is adapted and continuous, $f \in \mathfrak{I}$ by Lemma 4.2.4. Put $t_k^n = kt2^{-n}$. By Theorem 4.3.2, we find that $\sum_{k=1}^{2^n} f(t_{k-1}^n)(W_{t_k^n} - W_{t_{k-1}^n})$ converges in probability to $\int_0^t f(s)\,\mathrm{d}W_s$. However, the finite sequence of variables $W_{t_k^n} - W_{t_{k-1}^n}$ for $k = 1, \ldots, 2^n$ are independent and normally distributed with mean zero and variance $t2^{-n}$. Therefore, we find that $\sum_{k=1}^{2^n} f(t_{k-1}^n)(W_{t_k^n} - W_{t_{k-1}^n})$ is normally distributed with mean zero and variance $t2^{-n}\sum_{k=1}^{2^n} f(t_{k-1}^n)^2$. As f is continuous, this converges to $\int_0^t f(s)^2\,\mathrm{d}s$. Therefore, $\sum_{k=1}^{2^n} f(t_{k-1}^n)(W_{t_k^n} - W_{t_{k-1}^n})$ converges weakly to a normal distribution with mean zero and variance $\int_0^t f(s)^2\,\mathrm{d}s$. As this sequence of variables also converges in probability to $\int_0^t f(s)\,\mathrm{d}W_s$, and convergence in probability implies weak convergence, we conclude by uniqueness of limits that $\int_0^t f(s)\,\mathrm{d}W_s$ follows a normal distribution with mean zero and variance $\int_0^t f(s)^2\,\mathrm{d}s$. □

Solution to exercise 4.4.22. First note that by Theorem 4.3.5, $f(X)$ and $g(Y)$ are semimartingales, so the quadratic covariation is well-defined. By construction, we have

$$[f(X), g(Y)]_t = [f(X)^c, g(Y)^c]_t + \sum_{0<s\le t} \Delta f(X_s)\Delta g(Y_s).$$

Furthermore, by Theorem 4.3.5, we obtain

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_{s-})\,\mathrm{d}[X]_s + \eta_t,$$

up to indistinguishability, where η is in $\mathcal{V}$. From this, we see that the continuous martingale part of $f(X)$ is $f'(X_-)\cdot X^c$. Similarly, the continuous martingale part of $g(Y)$ is $g'(Y_-)\cdot Y^c$.

Applying Lemma 4.2.11, we thus obtain

$$\begin{aligned}
[f(X), g(Y)]_t &= [f'(X_-) \cdot X^c, g'(Y_-) \cdot Y^c]_t + \sum_{0<s\le t} \Delta f(X_s)\Delta g(Y_s) \\
&= \int_0^t f'(X_{s-})g'(Y_{s-})\,\mathrm{d}[X^c, Y^c]_s + \sum_{0<s\le t} \Delta f(X_s)\Delta g(Y_s) \\
&= \int_0^t f'(X_s)g'(Y_s)\,\mathrm{d}[X^c, Y^c]_s + \sum_{0<s\le t} \Delta f(X_s)\Delta g(Y_s),
\end{aligned}$$

where we also have used that $[X^c, Y^c]$ is continuous, so changing the integrand in countably many points does not change the value of the integral. This shows in particular that with W an $\mathcal{F}_t$ Brownian motion, $[W^p]_t = \int_0^t (pW_s^{p-1})^2\,\mathrm{d}[W]_s = p^2 \int_0^t W_s^{2(p-1)}\,\mathrm{d}s$. □

Solution to exercise 4.4.23. In the case where $i = j$, we may apply Itô's formula with the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and obtain $(W^i)_t^2 = 2\int_0^t W_s^i\,\mathrm{d}W_s^i + t$. Lemma 4.1.13 then shows that $[(W^i)^2]_t = 4[W^i \cdot W^i]_t = 4\int_0^t (W_s^i)^2\,\mathrm{d}s$. Next, consider the case where $i \neq j$. Applying Itô's formula with the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$, we have

$$W_t^i W_t^j = \int_0^t W_s^i\,\mathrm{d}W_s^j + \int_0^t W_s^j\,\mathrm{d}W_s^i.$$

Using Lemma 4.1.13 and Lemma 4.2.11, we then obtain

$$\begin{aligned}
[W^i W^j]_t &= [W^i \cdot W^j]_t + 2[W^i \cdot W^j, W^j \cdot W^i]_t + [W^j \cdot W^i]_t \\
&= \int_0^t (W_s^i)^2\,\mathrm{d}s + 2\int_0^t W_s^i W_s^j\,\mathrm{d}[W^i, W^j]_s + \int_0^t (W_s^j)^2\,\mathrm{d}s \\
&= \int_0^t (W_s^i)^2\,\mathrm{d}s + \int_0^t (W_s^j)^2\,\mathrm{d}s.
\end{aligned}$$

□

Solution to exercise 4.4.24. By Theorem 4.3.3, we know that $\sum_{k=1}^{2^n} (M_{t_k} - M_{t_{k-1}})^2$ converges to $[M]_t$ in probability. Therefore, by Lemma A.3.5, we have convergence in $\mathcal{L}^1$ if and only if the sequence of variables is uniformly integrable. To show uniform integrability, it suffices by Lemma A.3.4 to show boundedness in $\mathcal{L}^2$. To prove this, we first note the relationship $E(\sum_{k=1}^{2^n} (M_{t_k} - M_{t_{k-1}})^2)^2 = E\sum_{k=1}^{2^n} (M_{t_k} - M_{t_{k-1}})^4 + E\sum_{k\neq i} (M_{t_k} - M_{t_{k-1}})^2 (M_{t_i} - M_{t_{i-1}})^2$. With $C \geq 0$ being a constant such that $|M_t| \leq C$ for all $t \geq 0$, we may use the martingale property to obtain

$$
\begin{aligned}
E\sum_{k=1}^{2^n}(M_{t_k}-M_{t_{k-1}})^4 &\leq 4C^2\sum_{k=1}^{2^n}E(M_{t_k}-M_{t_{k-1}})^2\\
&= 4C^2\sum_{k=1}^{2^n}EM_{t_k}^2+EM_{t_{k-1}}^2-2EM_{t_k}M_{t_{k-1}}\\
&= 4C^2\sum_{k=1}^{2^n}EM_{t_k}^2-EM_{t_{k-1}}^2\leq 4C^4.
\end{aligned}
$$

Furthermore, we have by symmetry that

$$
E\sum_{k\neq i}(M_{t_k}-M_{t_{k-1}})^2(M_{t_i}-M_{t_{i-1}})^2=2E\sum_{k=1}^{2^n-1}\sum_{i=k+1}^{2^n}(M_{t_k}-M_{t_{k-1}})^2(M_{t_i}-M_{t_{i-1}})^2,
$$

and this is equal to $2E\sum_{k=1}^{2^n-1}(M_{t_k}-M_{t_{k-1}})^2\sum_{i=k+1}^{2^n}E((M_{t_i}-M_{t_{i-1}})^2|\mathcal{F}_{t_k})$, since M is adapted. Here, we may apply the martingale property to obtain

$$
\begin{aligned}
\sum_{i=k+1}^{2^n}E((M_{t_i}-M_{t_{i-1}})^2|\mathcal{F}_{t_k}) &= \sum_{i=k+1}^{2^n}E(M_{t_i}^2-2M_{t_i}M_{t_{i-1}}+M_{t_{i-1}}^2|\mathcal{F}_{t_k})\\
&= \sum_{i=k+1}^{2^n}E(M_{t_i}^2-M_{t_{i-1}}^2|\mathcal{F}_{t_k})=E(M_t^2-M_{t_k^n}^2|\mathcal{F}_{t_k})\leq C^2,
\end{aligned}
$$

which finally yields

$$
\begin{aligned}
E\sum_{k\neq i}(M_{t_k}-M_{t_{k-1}})^2(M_{t_i}-M_{t_{i-1}})^2 &\leq 2C^2E\sum_{k=1}^{2^n-1}(M_{t_k}-M_{t_{k-1}})^2\\
&= 2C^2\sum_{k=1}^{2^n-1}E(M_{t_k}^2-2M_{t_k}M_{t_{k-1}}+M_{t_{k-1}}^2)\\
&= 2C^2\sum_{k=1}^{2^n-1}EM_{t_k}^2-EM_{t_{k-1}}^2\leq 2C^4.
\end{aligned}
$$

Thus, we conclude $E(\sum_{k=1}^{2^n}(M_{t_k}-M_{t_{k-1}})^2)^2\leq 6C^4$, and so the sequence is bounded in $\mathcal{L}^2$. From our previous deliberations, we may now conclude that $\sum_{k=1}^{2^n}(M_{t_k}-M_{t_{k-1}})^2$ converges in $\mathcal{L}^1$ to $[M]_t$. □

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